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Multifractal Analysis of Choquet Capacities

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with respect to a general class of measures, and some preliminary results are presented concerning the usual spectra. In particular, we show how to construct a sequence of capacities whose Hölder spectrum is, under mild conditions, prescribed.

1. INTRODUCTION

Multifractal analysis was first introduced for the study of turbulence in [1–7] and was then much developed in a mathematical framework, for instance, in [8–18], where general results were obtained for deterministic or random *measures*. Other authors extended this analysis to *point functions* [19, 20], obtaining quite complete descriptions. In this work, a multifractal analysis is defined for *sequences of Choquet capacities* with respect to a given *reference measure*, and some preliminary results are given.

The motivations for these generalizations are essentially practical:

- In many applications, the relevant quantities for the description of a given phenomenon cannot be easily modeled by measures, because they are not additive. Let us give two examples. In the field of image analysis [21, 22], edge detection is a topical problem. To a given region in the image, we may associate the sum of the grey levels of the pixels lying inside this region. This set function is additive and it is a measure. Alternatively, we may split the region into subsets of pixels having the same grey level, and associate to the region the cardinal of the subset containing the largest number of elements. This set function is not additive; thus it is not

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a measure. However, it is increasing and regular; thus it is a Choquet capacity (a similar example is to be found in Section 5.1). Some computations show that the second set function is a more powerful and more robust tool for detecting edges in an image than the first set function. Let us now move to the study of road traffic. A topical quantity is the flow. It is usually expressed in number of vehicles per hour, and measures the number of vehicles that have been passing by at a given point during a given period of time. To a time interval $[t_0, t_1]$, we may associate the mean flow observed between t_0 and t_1, or the maximum flow observed between t_0 and t_1. While the former is additive and is a measure, the latter again is not additive and is a capacity. Furthermore, it is easy to see that the capacity is more appropriate than the measure if we are interested in predicting the beginning of a congestion.

These two simple examples show that additivity is too strong a requirement to impose on the set functions we deal with. Choquet capacities are the simplest generalization of measures relaxing the additivity constraint, and this is why it is interesting to define a multifractal analysis, such as that, for instance, presented in [14], does not indeed rely much on the fact that the studied objects are measures, and only a few details need to be modified in order to deal with capacities. Indeed, we closely follow [14] in Sections 2.1.4 and 3.3.

• There are many reasons for performing a multifractal analysis w.r.t. any probability measure μ rather than restricting to the Lebesgue measure. For instance, this generalization ‘‘uncovers’’ some degenerate cases: several measures may be mixed together with the singularities of one of them ‘‘dominating’’ the other so that only its spectrum is ‘‘seen’’ when we use a classical analysis. Changing the reference measure here allows one to be sensitive to the other measures. Another point is that a multifractal analysis may be meaningless or even impossible to perform w.r.t. the Lebesgue measure (for instance, the Hölder exponents may fail to exist), but can be fruitful w.r.t to another measure. Finally, and this is most important in practical applications, changing the reference measure may lead to much faster convergence rates when estimations are made on real data. We call this type of analysis *mutual multifractal analysis*, and a few steps are made in this direction in Section 3.6, although the subject is merely touched upon in the present paper. Further investigations will be presented elsewhere.

Finally, the reason for working with sequences is that the notion of resolution is taken into account in a simple manner. Also, most of our results need not insure the limit of the sequences.

Section 2 defines the basic principles of the analysis, Section 3 proposes some general results for the comparison of the different usual spectra and Section 4 proposes a simple model for constructing capacities which can be useful in some applications. The last section indicates a way to construct sequences of capacities with a prescribed spectrum, under some mild conditions, and give tighter bounds for the spectra.

2. MULTIFRACTAL ANALYSIS OF A SEQUENCE OF CHOQUET CAPACITIES

In this section, we define the quantities α_n, α, f_h, f_g, $\tilde{f}_g$ and f_l which are the core of multifractal analysis.

2.1. *General Definitions*

DEFINITION 1 [23]. Let E be a set. A paving on E is a set $\mathscr{E}$ of subsets of E containing the empty set and stable under finite union and finite intersection. The pair $(E, \mathscr{E})$ is called a paved space.

Let $\mathscr{P}(E)$ denote the power set of E.

DEFINITION 2. Let $(E, \mathscr{E})$ be a paved space. A Choquet $\mathscr{E}$ capacity on E is a function c: $\mathscr{P}(E) \to \overline{\mathbb{R}}$ verifying the following properties:

(1) c is non-decreasing: if $A \subset B$, then $c(A) \leq c(B)$.

(2) If (A_n) is an increasing sequence of subsets of E, i.e., $A_n \subseteq A_{n+1}$,

$$c\left(\bigcup_n A_n\right) = \sup_n c(A_n).$$

(3) If (A_n) is a decreasing sequence of elements of $\mathscr{E}$, i.e., $A_{n+1} \subseteq A_n$,

$$c\left(\bigcap_n A_n\right) = \inf_n c(A_n).$$

Remark. Every Borel measure can be extended to a capacity [23, p. 16], and every (positive) additive capacity such that $c(\emptyset) = 0$ is a measure on $\mathscr{B}(E)$ (the Borel sets of E). In what follows, we only consider Choquet capacities defined on $E := [0, 1[$, and taking values in $[0, 1]$, with $\mathscr{E} := \mathscr{B}(E)$. Moreover, the short term capacity will stand for a Choquet $\mathscr{E}$ capacity on E.

Let $c = (c_n)_{n \geq 1}$ be a sequence of capacities defined on $[0,1[$, and $\mathscr{P} := ((I_j^n)_{0 \leq j < \nu_n})_{n \geq 1}$ a sequence of partitions of $[0,1[$. We assume that the following conditions are met:

(C1) $\lim_{n \to \infty} \max_{0 \leq j < \nu_n} |I_j^n| + 0$.

(C2) $\forall n, k,\ I_k^n$ is an interval, semi-open to the right.

Occasionally, we will need the further conditions:

(C3) $\forall n, \forall j,\ 0 \leq j < \nu_n\ \ \exists k$ such that $I_j^n \subsetneqq I_k^{n-1}$, where $I_0^0 := E$.

(C4) $\forall \alpha > 0$,

$$\limsup_{I \in \mathscr{P},\, |I| \to 0} |I|^\alpha k(I) \leq 1,$$

where $k(I_j^n) := \sup\{|I_j^n|, |I_k^{n+1}|;\ I_k^{n+1} \subset I_j^n\}$.

We also assume that a non-atomic probability measure μ on $[0,1[$ is given.

We stress the fact that, in our case, a multifractal analysis is relative to a fixed sequence of partitions and a fixed measure. In particular, if the sequence of partitions changes, all the quantities defined below (i.e., $\alpha, f_h, f_l, f_g, \tilde{f}_g, \tau$) may vary.

For $x \in [0,1[$ and $n \in \mathbb{N}$, let $I^n(x)$ be the interval I_j^n containing x. Let U_n be the set of indices j such that $c_n(I_j^n)\mu(I_j^n)$ is strictly positive.

2.1.1. *Definition of f_h.* Let

$$\alpha_n(x) := \frac{\log c_n(I^n(x))}{\log \mu(I^n(x))},$$

which is defined when $c_n(I^n(x))\mu(I^n(x)) \neq 0$, and

$$\alpha(x) := \lim_{n \to \infty} \alpha_n(x)$$

when this limit exists. We call this quantity the pointwise Hölder exponent of c at point x with respect to μ, although the usual definition involves the limit over all balls centered at x, $c_n = c$ for all n, and $\mu = \mathscr{L}$ (Lebesgue measure).

We will use the following definition of dimension of a set E with respect to a non-atomic measure μ, $\dim_\mu(E)$. This definition is similar to that of Hausdorff dimension [24], except for the fact that it is restricted to coverings by the elements of $\mathscr{P}$.

Let

$$\mathscr{H}^s_{\mu,\delta}(E) := \inf\left\{ \sum_{i=1}^{+\infty} \mu(E_i)^s, E \subset \bigcup_i E_i, \mu(E_i) \le \delta, E_i \in \mathscr{P}\ \forall i \right\}$$

$$\mathscr{H}^s_\mu(E) := \lim_{\delta\to 0} \mathscr{H}^s_{\mu,\delta}(E)$$

$$\dim_\mu(E) := \inf\left\{s, \mathscr{H}^s_\mu(E) = 0\right\} = \sup\left\{s, \mathscr{H}^s_\mu(E) = +\infty\right\}.$$

Note that if the elements of $\mathscr{P}$ satisfy (C3) an d(C4), then $\dim_\mu(E)$ is indeed the Hausdorff dimension of E [25].

Set

$$E_\alpha := \{x \in [0,1[, \alpha(x) = \alpha\}.$$

The f_h multifractal spectrum (sometimes known as the Hölder or Hausdorff spectrum) of c is defined as

$$f_h(\alpha) := \dim_\mu E_\alpha .$$

2.1.2. *Definition of f_g.* Let $n \in \mathbb{N}$ and $\varepsilon > 0$. Let $K^n_\varepsilon(\alpha)$ and $N^n_\varepsilon(\alpha)$ denote

$$K^n_\varepsilon(\alpha) := \left\{ k \in \{0,\dots,\nu_{n-1}\}, \frac{c_n(I^n_k)}{\log \mu(I^n_k)} \in [\alpha-\varepsilon, \alpha+\varepsilon] \right\}$$

and

$$N^n_\varepsilon(\alpha) := \operatorname{card} K^n_\varepsilon(\alpha).$$

We define the f_g multifractal spectrum of c as

$$f_g(\alpha) := \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{\log N^n_\varepsilon(\alpha)}{\log \nu_n}.$$

Notice that, contrary to the usual definition of f_g [15, 26], we do not assume that all the intervals of a partition have the same size. ν_n does not represent here the inverse of the size of the intervals, but their number. However, this classical definition is obviously ill-adapted here. We are thus led to the following generalization.

2.1.3. *Definition of $\tilde{f}_g$.* With the previous notations, we define, for all $\beta > 0$,

$$S^n_\varepsilon(\alpha,\beta) := \sum_{k\in K^n_\varepsilon(\alpha)} \mu(I^n_k)^\beta$$

$$S_\varepsilon(\alpha,\beta) := \limsup_{n\to+\infty} S^n_\varepsilon(\alpha,\beta)$$

(with the convention $\sum_{\emptyset} = 0$). Using (C1), it is then easy to show, by analogy with the Hausdorff dimension, that there exists a real number $\tilde{f}_g^{\varepsilon}(\alpha)$ such that

$$\beta < \tilde{f}_g^{\varepsilon}(\alpha) \Rightarrow S_{\varepsilon}(\alpha, \beta) = +\infty$$

$$\beta > \tilde{f}_g^{\varepsilon}(\alpha) \Rightarrow S_{\varepsilon}(\alpha, \beta) = 0.$$

$\tilde{f}_g^{\varepsilon}$ is non-decreasing in ε, and we note that

$$\tilde{f}_g(\alpha) := \lim_{\varepsilon \to 0} \tilde{f}_g^{\varepsilon}(\alpha).$$

It is straightforward to verify that if all the intervals have the same size ν_n^{-1}, $\mu = \mathscr{L}$ and when f_g exists, then $f_g = \tilde{f}_g$ (see Lemma 1, below).

2.1.4. *Definition of* f_l. (Here we follow the work of [14].) Let $(\lambda_n)_{n \geq 1}$ be a sequence of positive integers such that

$$\sum_{n > 0} \exp(-\eta \lambda_n) < \infty \qquad \text{for all } \eta > 0. \tag{1}$$

We define

$$X_n(x, y) := \sum_{j \in U_n} c_n\left(I_j^n\right)^{x+1} \mu\left(I_j^n\right)^{-y}$$

and

$$X(x, y) := \limsup_{n \to \infty} \lambda_n^{-1} \log X_n(x, y).$$

One verifies that X is convex, non-decreasing in y, and non-increasing in x.

Set

$$\Omega := \{(x, y), X(x, y) < 0\}.$$

An argument similar to one found in [14] allows us to show that there exists a concave map ϕ such that

$$\mathring{\Omega} = \left\{(x, y) \in \mathbb{R}^2, y < \phi(x - 0)\right\}$$

($\mathring{\Omega}$ is the interior of Ω).

We suppose that ϕ is finite on an open interval containing 0, and we set

$$\tau(q) := \phi(q - 1).$$

We then define the f_l multifractal spectrum of c as the following Legendre transform of τ:

$$f_l(\alpha) := \inf_q [q\alpha - \tau(q)].$$

When $\mu = \mathscr{L}$ (Lebesgue measure) and all the I_j^n have the same length, say ν_n^{-1}, and assuming that all the considered limits exist, we obtain the usual formulae:

$$\tau_n(q) := -\frac{1}{\log \nu_n} \log \sum_{j \in U_n} c_n(I_j^n)^q$$

$$\tau(q) := \lim_{n \to \infty} \tau_n(q).$$

As for measures, the multifractal analysis consists in computing $\alpha_n(x)$ and $\alpha(x)$, and in evaluating and comparing f_h, f_g, $\tilde{f}_g$, and f_l.

When several sequences of capacities are considered simultaneously, we write respectively $E_{\alpha,c}$, $f_{h,c}$, $f_{g,c}$, $\tilde{f}_{g,c}$, and $f_{l,c}$ for E_α, f_h, f_g, $\tilde{f}_g$, and f_l associated with c, where c is either a measure or a sequence $c := (c_n)_{n \ge 1}$ of capacities.

2.2. *Remark*. When a sequence $(c_n)_n$ of capacities converges simply toward a set function c (i.e., for all $A \subset E$, $c_n(A)$ converges to $c(A)$), the definition of the Hölder exponent defined above might not coincide with the definition obtained by considering c (see the diagram below). More precisely, for a given $x \in E$, we could imagine the two following procedures for computing $\alpha(x)$.

(1) • Given c_n, compute $\alpha_n(x)$.
 • When the limit exists, deduce $\alpha(x) := \lim_{n \to +\infty} \alpha_n(x)$.

(2) • Compute $c := \lim_{n \to +\infty} c_n$.
 • Deduce $\alpha_c(x) := \lim_{n \to +\infty} (\log c(I^n(x)) / \log \mu(I^n(x)))$.

$$\begin{array}{ccc} c_n & \rightarrow & \alpha_n \\ \downarrow & & \downarrow \\ c & & \alpha \\ \downarrow & & \\ \alpha_c & & \end{array}$$

The following example shows that even in the case of a sequence of measures, we may have $\alpha_c \neq \alpha$:

Let x_0 be any element in $[0; 1[$, and $\mathscr{P}$ be the partition of $[0; 1[$ in dyadic intervals (i.e., $|I_k^n| = 2^{-n}$ for all k, n). The multifractal analysis is here carried out with respect to $\mu = \mathscr{L}$. We consider the sequence of measures whose general term is

$$c_n(A) := \mathscr{L}(A \setminus I^n(x_0)) + 2^{-n} \mathscr{L}(A \cap I^n(x_0))$$

for every Borel set A of E. Clearly, $c_n(A)$ converges toward $\mathscr{L}(A)$. Moreover, $c_n(I^n(x_0)) = 2^{-2n}$, and for $x \neq x_0$, and for sufficiently large n, $c_n(I^n(x)) = 2^{-n}$. We deduce

$$\alpha(x_0) = 2$$

$$\alpha(x) = 1 \qquad \text{if } x \neq x_0$$

$$\alpha_c(x) = 1 \qquad \text{for all } x \in E.$$

A necessary and sufficient condition that guarantees the commutativity of the above diagram is of course

$$\lim_{n \to +\infty} \frac{1}{n} \log \frac{c_n(I^n(x))}{c(I^n(x))} = 0.$$

However, in practice, the limit is not always known. The following commutativity criterion is more convenient:

PROPOSITION 1. *Let (c_n) be a sequence of Choquet capacities converging simply to a finite limit c, and let $x \in E$ such that $\alpha(x)$ and $\alpha_c(x)$ exist. Then*

$$\left(\lim_{n \to +\infty} \lim_{p \to +\infty} \frac{1}{n} \log \frac{c_n(I^n(x))}{c_p(I^n(x))} = 0 \right) \Leftrightarrow \left(\alpha_c(x) = \alpha(x) \right).$$

If so, c_n is said to be an admissible rank n approximation of c.

Proof. Use the equality $c(I^n(x)) = \lim_{p \to +\infty} c_p(I^n(x))$ for a fixed n. ∎

The measure introduced in the example does not satisfy the commutativity criterion. Indeed, we have

$$\lim_{n \to +\infty} \lim_{p \to +\infty} \frac{1}{n} \log \frac{\mu_n(I^n(x_0))}{\mu_p(I^n(x_0))} = -\log 2.$$

3. COMPARISON OF $f_h, f_g, \tilde{f}_g, f_l$ IN THE GENERAL CASE

In this section, we propose some inequalities between f_h, f_g, $\tilde{f}_g$, and f_l without making any assumption on c. The main result of this section is the following:

THEOREM 1. *Let $c := (c_n)_{n \geq 1}$ be a sequence of Choquet $\mathscr{B}(E)$-capacities defined on $[0, 1[$, taking values in $[0, 1]$, and let $\mathscr{P} := ((I_k^n)_{0 \leq k < \nu_n})_{n \geq 1}$ be a sequence of partitions of $[0, 1[$ satisfying (C1) and (C2). Then the following inequalities hold*:

$$f_h \leq \tilde{f}_g \leq f_l.$$

The proof of this theorem is separated into several steps explained below.

3.1. *Comparison of* f_h *and* $\tilde{f}_g$

PROPOSITION 2. *Under conditions* (C1) *and* (C2), *we have*

$$f_h \le \tilde{f}_g .$$

Proof. In what follows, we assume that $E_\alpha \neq \emptyset$ (the case $E_\alpha = \emptyset$ is trivial). Let $x \in [0;1[$, $n \in \mathbb{N}$, $k_n(x)$ the integer such that $x \in I^n_{k_n(x)}$, and x^n_k the lower bound of I^n_k. When the limit exists,

$$\alpha(x) := \lim_{n \to +\infty} \alpha_n(x) = \lim_{n \to +\infty} \alpha_n\left(x^n_{k_n(x)}\right).$$

Choose $\varepsilon > 0$. We have

$$\exists n_0 = n_0(x, \varepsilon), \forall n \ge n_0 \qquad \alpha_n\left(x^n_{k_n(x)}\right) \in [\alpha - \varepsilon; \alpha + \varepsilon]$$

or equivalently

$$\exists n_0 = n_0(x, \varepsilon), \forall n \ge n_0 \qquad k_n(x) \in K^n_\varepsilon(\alpha).$$

Since $x \in I^n_{k_n(x)}$ for all n, we obtain

$$\forall \varepsilon > 0 \qquad x \in \bigcap_{n \ge n_0(x,\varepsilon)} \bigcup_{k \in K^n_\varepsilon(\alpha)} I^n_k$$

and

$$\forall \varepsilon > 0 \qquad E_\alpha \subset \bigcup_p \bigcap_{n \ge p} \bigcup_{k \in K^n_\varepsilon(\alpha)} I^n_k ,$$

which yields, after we set $E^\varepsilon_{\alpha,p} := \bigcap_{n \ge p} \bigcup_{k \in K^n_\varepsilon(\alpha)} I^n_k$ and $s^\varepsilon_p := \dim_\mu E^\varepsilon_{\alpha,p}$,

$$f_h(\alpha) := \dim_\mu E_\alpha \le \sup_p s^\varepsilon_p .$$

Clearly, $E^\varepsilon_{\alpha,p} \subset E^\varepsilon_{\alpha,p+1}$, and thus $s^\varepsilon_p \le s^\varepsilon_{p+1}$. We deduce $\sup_p s^\varepsilon_p = \lim_{p \to +\infty} s^\varepsilon_p$ (which is finite since $s^\varepsilon_p \le 1$).

Let $s, \delta \in \mathbb{R}^+$ and $p \in \mathbb{N}$. For large n, we have $\max_k \mu(I^n_k) \le \delta$ and thus

$$\mathscr{H}^s_{\mu,\delta}\left(E^\varepsilon_{\alpha,p}\right) \le \sum_{k \in K^n_\varepsilon(\alpha)} \mu(I^n_k)^s = S^n_\varepsilon(\alpha, s),$$

which yields

$$\mathscr{H}^s_\mu\left(E^\varepsilon_{\alpha,p}\right) \le S_\varepsilon(\alpha, s)$$

and

$$s^\varepsilon_p \le \tilde{f}^\varepsilon_g(\alpha).$$

We conclude that

$$f_h(\alpha) \leq \tilde{f}_g(\alpha). \quad \blacksquare$$

3.2. *Comparison of $\tilde{f}_g$ and f_l*

PROPOSITION 3. *Under conditions* (C1) *and* (C2), *we have*

$$\tilde{f}_g \leq f_l.$$

Proof. Since

$$X_n(q-1,\tau) := \sum_{k=0}^{\nu_n - 1} c_n(I_k^n)^q \mu(I_k^n)^{-\tau} \geq \sum_{k \in K_\varepsilon^n(\alpha)} c_n(I_k^n)^q \mu(I_k^n)^{-\tau}$$

we have, for large n,

$$X_n(q-1,\tau) \geq \sum_{k \in K_\varepsilon^n(\alpha)} \mu(I_k^n)^{q(\alpha \pm \varepsilon)-\tau} =: S_\varepsilon^n(\alpha,(\alpha \pm \varepsilon)q - \tau)$$

(choose $\alpha + \varepsilon$ if $q \geq 0$ and $\alpha - \varepsilon$ if $q < 0$).

Choose $\tau < \tau(q)$. Then $X(q-1,\tau) < 0$ and there exists $c > 0$ such that, for large n,

$$X_n(q-1,\tau) \leq \exp(-c\lambda_n).$$

This yields

$$S_\varepsilon(\alpha,(\alpha \pm \varepsilon)q - \tau) = 0.$$

Hence

$$\forall q, \forall \tau < \tau(q), \forall \varepsilon > 0 \qquad (\alpha \pm \varepsilon)q - \tau \geq \tilde{f}_g^\varepsilon(\alpha).$$

Thus, letting $\varepsilon \to 0$ and $\tau \to \tau(q)$,

$$\forall q \qquad \tilde{f}_g(\alpha) \leq \alpha q - \tau(q)$$

and

$$\tilde{f}_g(\alpha) \leq f_l(\alpha). \quad \blacksquare$$

3.3. *Comparison of f_h and f_l*

This section follows some results of [14] to show that, in the case of a sequence of Choquet capacities we have:

PROPOSITION 4. *Under conditions* (C1) *and* (C2),

$$f_h \leq f_l.$$

Proof. The proof is a simple extension of the one given in [14]. Here are the main steps:

- We show that $f_l(\alpha) = \inf_q(q\alpha - \tau(q))$ is non-decreasing on $]-\infty, -\tau'(0^+)]$ and non-increasing on $[\tau'(0^-), +\infty[$.
- If $\alpha \le \tau'(0^-)$ and $\delta > f_l(\alpha)$, then $\exists t > 0, X(t-1, -\delta + \alpha t) < 0$.
- If $\alpha \ge \tau'(0^+)$ and $\delta > f_l(\alpha)$, then $\exists t > 0, X(-t-1, -\delta - \alpha t) < 0$.
- Let $\overline{U}_\alpha = \{j, c_n(I_j^n) \ge \mu(I_j^n)^\alpha\}$; then $\limsup_{n\to\infty} \lambda_n^{-1} \log \sum_{j\in\overline{U}_\alpha} \mu(I_j^n)^\delta < 0$ for $\alpha \le \tau'(0^-)$ and $\delta > f_l(\alpha)$.
- Let $\underline{U}_\alpha = \{j, 0 < c_n(I_j^n) \le \mu(I_j^n)^\alpha\}$, then $\limsup_{n\to\infty} \lambda_n^{-1} \log \sum_{j\in\underline{U}_\alpha} \mu(I_j^n)^\delta < 0$ for $\alpha \ge \tau'(0^+)$ and $\delta > f_l(\alpha)$.
- If $\alpha < \tau'(0^+)$, then $\limsup_{n\to\infty} \lambda_n^{-1} \log \sum_{j\in\overline{U}_\alpha} \mu(I_j^n)^{-\tau(0)} < 0$.
- If $\alpha > \tau'(0^-)$, then $\limsup_{n\to\infty} \lambda_n^{-1} \log \sum_{j\in\underline{U}_\alpha} \mu(I_j^n)^{-\tau(0)} < 0$.
- Let $B_\alpha = \{x \in [0,1[\mid \alpha(x) \le \alpha\}$. If $\alpha \le \tau'(0^-)$, then $\dim_\mu B_\alpha \le f_l(\alpha)$. (X)
- Let $V_\alpha = \{x \in [0,1[\mid \alpha(x) \ge \alpha\}$. If $\alpha \ge \tau'(0^+)$, then $\dim_\mu V_\alpha \le f_l(\alpha)$. (Y)

This finally leads to

$$f_h(\alpha) = \dim_\mu E_\alpha \le f_l(\alpha). \qquad \blacksquare (Z)$$

Remark. Results (X) and (Y) are in fact stronger than (Z), since they provide an upper bound for the dimension of the union of a possibly uncountable number of sets E_α.

3.4. *Comparison of $\tilde{f}_g$ and f_g*

Let c and ε be two strictly positive real numbers, and n an integer. Set, for all $\alpha \in \mathbb{R}^+$,

$$\underline{K}_\varepsilon^n(\alpha, c) := \left\{k \in K_\varepsilon^n(\alpha), \mu(I_k^n) < c\nu_n^{-1}\right\}$$

$$\overline{K}_\varepsilon^n(\alpha, c) := \left\{k \in K_\varepsilon^n(\alpha), \mu(I_k^n) > c\nu_n^{-1}\right\}$$

$$\underline{\eta}_\varepsilon^n(\alpha, c) := \operatorname{card} \underline{K}_\varepsilon^n(\alpha, c)$$

$$\overline{\eta}_\varepsilon^n(\alpha, c) := \operatorname{card} \overline{K}_\varepsilon^n(\alpha, c)$$

$$N_\varepsilon^n(\alpha) := \operatorname{card} K_\varepsilon^n(\alpha)$$

$$l_\varepsilon^n(\alpha) := \min_{k\in K_\varepsilon^n(\alpha)} \mu(I_k^n)$$

$$\lambda_\varepsilon^n(\alpha) := \max_{k\in K_\varepsilon^n(\alpha)} \mu(I_k^n).$$

Furthermore, we note, for all $\beta > 0$, that

$$m_n^\varepsilon(\alpha, c, \beta) := \frac{\underline{\eta}_\varepsilon^n(\alpha, c)}{N_\varepsilon^n(\alpha)}\left(1 - \left(\frac{l_\varepsilon^n(\alpha)}{c\nu_n^{-1}}\right)^\beta\right)$$

$$M_n^\varepsilon(\alpha, c, \beta) := \frac{\overline{\eta}_\varepsilon^n(\alpha, c)}{N_\varepsilon^n(\alpha)}\left(\left(\frac{\lambda_\varepsilon^n(\alpha)}{c\nu_n^{-1}}\right)^\beta - 1\right).$$

From the definition of $S_\varepsilon^n(\alpha, \beta)$, we deduce the inequalities

$$\begin{aligned} c^\beta N_\varepsilon^n(\alpha)\nu_n^{-\beta}(1 - m_n^\varepsilon(\alpha, c, \beta)) &\leq S_\varepsilon^n(\alpha, c) \\ &\leq c^\beta N_\varepsilon^n(\alpha)\nu_n^{-\beta}(1 + M_n^\varepsilon(\alpha, c, \beta)). \end{aligned}$$

We then obtain the following lemma:

LEMMA 1. *Under conditions* $(C1)$ *and* $(C2)$, *we have*

1. $(\exists c > 0, \forall \varepsilon > 0, \ \forall \beta < f_g(\alpha), \ \limsup_n m_n^\varepsilon(\alpha, c, \beta) < 1) \Rightarrow f_g(\alpha) \leq \tilde{f}_g(\alpha)$.
2. $(\exists c > 0, \forall \varepsilon > 0, \ \forall \beta > f_g(\alpha), \ \limsup_n M_n^\varepsilon(\alpha, c, \beta) < +\infty) \Rightarrow \tilde{f}_g(\alpha) \leq f_g(\alpha)$.
3. $(\exists A > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \ A^{-1}\nu_n^{-1} \leq \mu(I_k^n) \leq A\nu_n) \Rightarrow f_g = \tilde{f}_g$.

Proof. 1. The hypothesis implies that there exists a real number $m \in]0; 1[$ such that

$$m_n^\varepsilon(\alpha, c, \beta) < m$$

for all large n, and thus

$$c^\beta(1 - m)N_\varepsilon^n(\alpha)\nu_n^{-\beta} \leq S_n^\varepsilon(\alpha, \beta)$$

and the result follows.

2. The proof of the second result is similar.

3. This is a consequence of the two other results. Indeed,

$$A^{-1}\nu_n^{-1} \leq \mu(I_k^n) \Rightarrow \underline{\eta}_\varepsilon^n(\alpha, A^{-1}) = 0$$

$$\mu(I_k^n) \leq A\nu_n^{-1} \Rightarrow \overline{n}_\varepsilon^n(\alpha, A) = 0,$$

which yields the desired equality. ∎

We also have more practical properties:

PROPOSITION 5. 1. $(\exists c > 0, \forall \varepsilon > 0, \ \limsup_n (\underline{\eta}_\varepsilon^n(\alpha, c)/N_\varepsilon^n(\alpha)) < 1) \Rightarrow f_g(\alpha) \leq \tilde{f}_g(\alpha)$.

2. $(\forall \varepsilon > 0, \liminf_n l_\varepsilon^n(\alpha)\nu_n > 0) \Rightarrow f_g(\alpha) \leq \tilde{f}_g(\alpha)$.

3. $(\exists c > 0, \forall \varepsilon > 0, \limsup_n \underline{\eta}_\varepsilon^n(\alpha, c) < +\infty) \Rightarrow f_g(\alpha) \leq \tilde{f}_g(\alpha)$.

4. $(\forall \varepsilon > 0, \limsup_n \lambda_\varepsilon^n(\alpha)\nu_n < +\infty) \Rightarrow \tilde{f}(\ \leq f_g(\alpha)$.

5. $(\exists c > 0, \forall \varepsilon > 0, \limsup_n \overline{\eta}_\varepsilon^n(\alpha, c) < +\infty) \Rightarrow \tilde{f}_g(\alpha) \leq f_g(\alpha)$.

Proof. 1. $\limsup_n (\eta_\varepsilon^n(\alpha, c)/N_\varepsilon^n(\alpha)) < 1 \Rightarrow \underline{\eta}_\varepsilon^n(\alpha, c)/N_\varepsilon^n(\alpha) < m < 1$ for a certain m and for large n, and

$$\limsup_n m_n^\varepsilon(\alpha, c, \beta) < 1.$$

We conclude by using the previous lemma.

2. $$\left(\liminf_n l_\varepsilon^n(\alpha)\nu_n > 0\right) \Rightarrow (\exists c > 0 \mid l_\varepsilon^n(\alpha)\nu_n > c) \Rightarrow \underline{\eta}_\varepsilon^n(\alpha, c) = 0.$$

3. We have

$$c^\beta N_\varepsilon^n(\alpha)\nu_n^{-\beta}\underline{\eta}_\varepsilon^n(\alpha, c)\left(\nu_n^{-\beta} - l_\varepsilon^n(\alpha)^\beta\right) \leq S_\varepsilon^n(\alpha, c)$$

and $\limsup_n \underline{\eta}_\varepsilon^n(\alpha, c) < +\infty$ ensures that $\underline{\eta}_\varepsilon^n(\alpha, c)$ is bounded. Moreover,

$$\lim_n \left(\nu_n^{-\beta} - l_\varepsilon^n(\alpha)^\beta\right) = 0$$

implies that $\underline{\eta}^n \alpha, c)(\nu_n^{-\beta} - l_\varepsilon^n(\alpha)^\beta)$ is bounded.

4. The proof is similar to that of part 3.

5. The proof is similar to that of part 3.

3.5. *Examples When f_h, f_g, and f_l Differ with $\mu = \mathscr{L}$*

3.5.1. EXAMPLE 1. $f_g \leq f_h$. In this example, the sequence $(c_n)_n$ is such that, for all n, $c_n := \overline{\mu}$ with $\overline{\mu} := \frac{1}{2}(\mu_1 + \mu_2)$, where μ_1 is a binomial measure[1] whose support is $[0; \frac{1}{2}[$, with weights $m_0, m_1 := 1 - m_0$ $(m_0 < m_1)$, and μ_2 is the uniform measure on $[\frac{1}{2}; 1[$. We split $[0; 1[$ as follows: for a given $n \in \mathbb{N}^*$, we split $[0; \frac{1}{2}[$ in dyadic intervals of size $2^{-(n-1)}$, and $[\frac{1}{2}; 1[$ in triadic intervals of size $3^{-(n-1)}$. For almost every $x \in [0; \frac{1}{2}[$, $\alpha(x) = -\varphi_0 \log_2 m_0 - \varphi_1 \log_2 m_1$. For any $x \in [\frac{1}{2}; 1[$, $\alpha(x) = 1$. Clearly,

$$f_h(\alpha) = \begin{cases} -\varphi_0 \log_2 \varphi_0 - \varphi_1 \log_2 \varphi_1 & \text{if } \alpha \in [-\log_2 m_1; -\log_2 m_0] \setminus \{1\} \\ 1 & \text{if } \alpha = 1. \end{cases}$$

[1] For the definition and results on multinomial measures, see, for instance, [14].

But it is easily shown that

$$f_g(\alpha) = \begin{cases} -\varphi_0 \log_3 \varphi_0 - \varphi_1 \log_3 \varphi_1 & \text{if } \alpha \in [-\log_2 m_1; -\log_2 m_0] \setminus \{1\} \\ 1 & \text{if } \alpha = 1. \end{cases}$$

This example shows that, in the case of a non-uniform partition, f_g is inadequate. Indeed, one easily verifies that, in this case, $f_h(\alpha) = \tilde{f}_g(\alpha)\ \forall \alpha$.

3.5.2. EXAMPLE 2. $f_h = f_g \leq f_l$. Here again, for all n, $c_n := \overline{\mu}$ defined on $[0; 1[$ by $\overline{\mu} := \frac{1}{2}(\mu_1 + \mu_2)$, where μ_1 is the uniform measure on $[0; \frac{1}{2}[$, and μ_2 is a binomial measure on $[\frac{1}{2}; 1[$ with parameters m_0, m_1, and the partition of $[0; 1[$ is made up of dyadic intervals.

In this example, we shall refer to indices 1 and 2 when talking about quantities associated with the measures μ_1 and μ_2, respectively. Thus, we denote by f_1 and f_2 the spectra associated with μ_1 and μ_2, respectively, and we denote by $\overline{\alpha}$ the real number such that $\overline{\alpha} = f_{g,2}(\overline{\alpha})$ and by α_M the real number such that $f_{g,2}(\alpha_M) = 1$.

f_h spectrum. Since μ_1 and μ_2 have disjoint supports, we have

$$E_\alpha = E_\alpha^{\mu_1} \cup E_\alpha^{\mu_2}$$

and

$$f_h(\alpha) = \max\left(\dim_H E_\alpha^{\mu_1}, \dim_H E_\alpha^{\mu_2}\right)$$

with

$$\dim_H E_\alpha^{\mu_1} = \begin{cases} -\infty & \text{if } \alpha \neq 1 \\ 1 & \text{if } \alpha = 1 \end{cases}$$

and

$$\dim_H E_\alpha^{\mu_2} = -\varphi_0 \log_2 \varphi_0 - \varphi_1 \log_2 \varphi_1 \leq 1,$$

which gives

$$f_h(\alpha) = \begin{cases} 1 & \text{if } \alpha = 1 \\ f_2(\alpha) & \text{if } \alpha \neq 1 \end{cases}$$

(notice that f_h is not concave).

f_g spectrum. A simple computation shows that $f_g(1) = 1$ and $f_g(\alpha) = -\varphi_0 \log_2 \varphi_0 - \varphi_1 \log_2 \varphi_1$ if $\alpha \neq 1$. In this case, we have the equality

$$f_h = f_g.$$

f_l spectrum. For a given n, we have

$$\tau_n(q) = \frac{q}{n} - \frac{1}{n} \log_2\left((2^{1-q})^{n-1} + (m_0^q + m_1^q)^{n-1} \right)$$

and

$$f_l(\alpha) = \min\Big(\inf_{q\in[0;1]} \{\alpha q - q + 1\};\ \inf_{q\le 0} \{\alpha q - \tau(q)\};\ \inf_{q\ge 1} \{\alpha q - \tau(q)\}\Big).$$

Since

$$\inf_{q\in[0;1]} \{\alpha q - q + 1\} = \min(\alpha, 1)$$

$$\inf_{q\le 0} \{\alpha q - \tau(q)\} = \begin{cases} 1 & \text{if } \alpha \le \alpha_M \\ f_g(\alpha) & \text{if } \alpha \ge \alpha_M \end{cases}$$

$$\inf_{q\ge 1} \{\alpha q - \tau(q)\} = \begin{cases} f_g(\alpha) & \text{if } \alpha \le \overline{\alpha} \\ \alpha & \text{if } \alpha \ge \overline{\alpha}. \end{cases}$$

We obtain:

- If $\alpha \in [-\log_2 m_1; \overline{\alpha}]$ (and then $\alpha < 1 < \alpha_M$)

$$f_l(\alpha) = \min\big(\min(\alpha, 1), 1, f_g(\alpha)\big) = f_g(\alpha).$$

- If $\alpha \in [\overline{\alpha}; \alpha_M]$,

$$f_l(\alpha) = \min(\min(\alpha, 1), 1, \alpha) = \min(\alpha, 1) > f_g(\alpha) = f_h(\alpha).$$

- If $\alpha \in [\alpha_M; -\log_2 m_0]$ (and thus $\alpha > 1 > \overline{\alpha}$),

$$f_l(\alpha) = \min\big(\min(\alpha, 1), f_g(\alpha), \alpha\big) = f_g(\alpha).$$

3.5.3. EXAMPLE 3. $f_h \le f_g$. For $n \in \mathbb{N}^*$, we split $[0;1[$ in dyadic intervals of size 2^{-n}. We consider the sequence of probability measures indexed by n,

$$\mu_n(A) := \frac{1}{2}\Bigg[\mathscr{L}(A) + \sum_{p\ge 1} 2^{-p}\delta_{1/p}(A)\Bigg].$$

For a Borel subset A of $[0,[$, set $P := \{1/p;\ p \in \mathbb{N}^*\}$. Let $x \in [0;1[$. One easily checks that

$$\alpha(x) = \begin{cases} 0 & \text{if } x \in P \\ 1 & \text{if } x \notin P. \end{cases}$$

Therefore,

$$f_h(0) = \dim_{\mathscr{L}} P = 0$$
$$f_h(1) = 1.$$

Computing f_g gives

$$f_g(0) = \frac{1}{2}$$
$$f_g(1) = 1.$$

Thus

$$f_h(0) < f_g(0).$$

Another example in which f_g and $\tilde{f}_g$ differ dramatically on $[0, 1[$ is presented in Section 5.3, Example 4.

3.6. *A Simple Example with* $\mu \neq \mathscr{L}$

In this section, we present an explicit computation in a case where $\mu \neq \mathscr{L}$. However, as was said in the Introduction, we will not elaborate much on this topic here, and a full account of the study of multifractal analysis with respect to ‘‘exotic’’ measures will be presented elsewhere.

We analyze the binomial measure ν on $[0; 1[$, with weights (m_0, m_1) with respect to another binomial measure μ on $[0; 1[$, whose weights are (p_0, p_1), the partition being the dyadic intervals. In this case, we compute f_h and f_l, and show that they are equal, which implies that their common value is also the value of $f_g(\alpha) = \tilde{f}_g(\alpha)$. Thus, the so-called ‘‘multifractal formalism’’ (i.e., $f_h = f_g = f_l$) still holds in the case of binomial measures analyzed with respect to other binomial measures.

Computation of α_μ. To emphasize the dependence on μ, we shall write $\alpha_\mu(x)$ instead of $\alpha(x)$ for the Hölder exponent of ν at x with respect to μ. Let $x \in [0; 1[$ be such that $\varphi_0 := \varphi_0(x)$ exists, where $\varphi_0(x)$ is the proportion of zeros in the dyadic expansion of x. A straightforward computation gives

$$\alpha_\mu(x) = \frac{-\varphi_0 \log m_0 - \varphi_1 \log m_1}{-\varphi_0 \log p_0 - \varphi_1 \log p_1}.$$

Computation of f_h. Let us define the notations

$$\alpha = \log(1/p_1), \qquad b = -\log(1/m_1),$$
$$c = \log(p_0/p_1), \qquad d = -\log(m_0/m_1)$$
$$D := \{x \in [0,1[, \varphi_0(x) \text{ exists}\}$$
$$E(\varphi) := \{x \in D, \varphi_0(x) = \varphi\}$$
$$L(\alpha) := \{x \in D, \alpha_{\mathscr{L}}(x) = \alpha_{\mathscr{L}}\}$$
$$M(\alpha_\mu) := \{x \in D, \alpha_\mu(x) = \alpha_\mu\},$$

where $\alpha_{\mathscr{L}}(x)$ represents the Hölder exponent of ν at x with respect to $\mathscr{L}$. It is easy to see that

$$E(\varphi) = M\left(\frac{-\varphi d + b}{\varphi c - \alpha}\right) = L(\varphi d - b).$$

Using the Kinney–Pitcher–Billingsley theorem [24, 14.1, p. 141; 3], we get

$$\alpha_\mu = \frac{-\varphi_0 \log m_0 - \varphi_1 \log m_1}{-\varphi_0 \log p_0 - \varphi_1 \log p_1}$$

$$f_h(\alpha_\mu) = \dim_\mu M(\alpha_\mu) = \frac{-\varphi_0 \log \varphi_0 - \varphi_1 \log \varphi_1}{-\varphi_0 \log p_0 - \varphi_1 \log p_1}.$$

Here, the f_h spectrum has the familiar bell shape observed usually for multinomial measures, with maximum value equal to 1 and the line $y = x$ tangent to the graph. Note however that, contrary to the classical case, the spectrum is not symmetric in general. Of course, if $p_0 = p_1 = \frac{1}{2}$, then $\mu = \mathscr{L}$, and we recover the well known result for f_h:

$$f_h(\alpha_{\mathscr{L}}) = -\varphi_0 \log_2 \varphi_0 - \varphi_1 \log_2 \varphi_1.$$

Another obvious limit case is the one where we analyze the measure ν with respect to itself, i.e., $\mu = \nu$. Here, α_μ always equals 1.

Note that, in general, the "spectrum" with the Hausdorff dimension computed with respect to $\mathscr{L}$ need not be concave. In other words, the mapping $\alpha_\mu \mapsto \dim_{\mathscr{L}} M(\alpha_\mu)$ is not concave as soon as p_0 and m_0 differ sufficiently.

Computation of f_l. It is easy to see that $\tau(q)$ is given by the implicit formula

$$m_0^q p_0^{-\tau(q)} + m_1^q p_1^{-\tau(q)} = 1.$$

Even though we cannot in general derive an explicit expression for τ (except of course when $p_0 = p_1$), it is possible to obtain such a formula for f_l. Indeed, the Legendre transform of τ is easily computed to be, in a parametric form,

$$\alpha_\mu(\varphi) = \frac{-\varphi \log m_0 - (1-\varphi)\log m_1}{-\varphi \log p_0 - (1-\varphi)\log p_1}$$

$$f_l(\varphi) = q(\varphi)\,\alpha(\varphi) - \tau(\varphi)$$

with

$$q(\varphi) = \frac{\log(1-\varphi)\log p_0 - \log\varphi \log p_1}{\log m_1 \log p_0 - \log m_0 \log p_1}$$

$$\tau(\varphi) = \frac{\log(1-\varphi)\log m_0 - \log\varphi \log m_1}{\log m_1 \log p_0 - \log m_0 \log p_1}.$$

We verify that

$$f_h(\alpha_\mu) = f_l(\alpha_\mu).$$

Comments. These computations can easily be extended to multinomial measures. As an application, consider the two following measures:

μ_1: trinomial measure on [0; 1[, with weights m_0, m_1, m_2

μ_2: quadrinomial measure on [0; 1[, with weights p_0, p_1, p_2, p_3.

We take

$$p_0 = m_0^{\log 4/\log 3}, \qquad p_3 = m_2^{\log 4/\log 3}, \qquad m_0 < m_1 < m_2$$

$$p_1 = \frac{1-(p_0+p_3)}{2} + \varepsilon, \qquad p_2 = \frac{1-(p_0+p_3)}{2} - \varepsilon, \qquad 0 < \varepsilon \ll 1.$$

Then, if the analysis is performed with respect to the Lebesgue measure, we have that (the indices refer to the measures)

$$\tau_1(q) = \tau_2(q) \qquad \text{for } q \in \{-\infty; 0; 1; +\infty\}.$$

If, for instance, we choose $m_0 = 0.1$, $m_1 = 0.4$, $\varepsilon = \frac{1}{11}$, then the relative "error," i.e.,

$$\max\left(\left|\frac{\tau_1(q) - \tau_2(q)}{\tau_1(q)}\right|, \left|\frac{\tau_1(q) - \tau_2(q)}{\tau_2(q)}\right|\right)$$

is always smaller that 0.006. As a consequence, numerical estimations of $(q, \tau(q))$ or $(\alpha, f_g(\alpha))$ will never be able to distinguish the two measures even if we are dealing with a huge number of data. This example shows that, for practical purposes, we do need other methods for finely estimating the multiplicative properties of even simple multinomial measures: a solution, in the case presented above, would be to perform the analysis with respect to μ_1, enabling the two spectra to clearly differ.

Let us mention briefly another example. Let μ_1 be the trinomial measure with weights (m_0, m_1, m_2) on $[2, 3[$, with $m_0 = m_2 = 0.1$, $m_1 = 0.8$, and μ_2 the binomial measure on the triadic Cantor set $\mathscr{C} \subset [0; 1]$ with weights $(p, 1-p)$, where p is chosen such that

$$\tau_1'(1) = \tau_2'(1)$$

(this equation has two solutions with the chosen numerical values), with

$$\tau_1(q) = -\log_3(m_0^q + m_1^q + m_3^q)$$

$$\tau_2(q) = -\log_3\big(p^q + (1-p)^q\big).$$

In our case, p approximately equals 0.68. It is easy to verify that, denoting by f_1 (resp. f_2) the f_h spectrum of μ_1 (resp. μ_2),

$$\forall \alpha, \qquad f_2(\alpha) \leq f_1(\alpha).$$

Since μ_1 and μ_2 have disjoint supports, the f_h spectrum of $\mu = \frac{1}{2}(\mu_1 + \mu_2)$ with respect to $\mathscr{L}$ is

$$f_h(\alpha) = \max\big(f_1(\alpha), f_2(\alpha)\big) = f_1(\alpha).$$

Here, the singularities coming from μ_2 are "hidden" by those generated by μ_1. This is another case where a classical multifractal analysis fails to correctly describe the multifractal properties of a measure μ. On the contrary, a multifractal analysis with respect to μ_1 would allow us to uncover these properties.

Of course, when we are facing a real situation, the question is: How do we choose adequately the reference measures? This remains an open problem at this stage.

4. CONSTRUCTION OF CAPACITIES

In this section, we define particular classes of Choquet capacities which allow us to take into account in a simple manner the notation of resolution.

4.1. *Myopic Capacities*

Let $E \subset \mathbb{R}$, $(D_n)_{n \geq 1}$ be a sequence of finite subsets of $\mathbb{N}$ such that $\nu_n := \text{card}\, D_n \to +\infty$ with n. Let $X_n := \{x_k^n \mid k \in D_n\}$ be a sequence of finite sets of distinct points x_k^n of E, and $\mathscr{P} := ((I_k^n)_{0 \leq k < \nu_n})_{n \geq 1}$ be a sequence of partitions of E such that, for all $n \geq 1$, $k \in D_n$, each I_k^n contains exactly one x_k^n.

For all $A \in \mathscr{P}(E)$, we note that

$$K_n(A) = \{k \in D_n, x_k^n \in A\}.$$

If we call i_n the one-to-one map from D_n into X_n, we can write

$$K_n(A) = i_n^{-1}(A \cap X_n).$$

We deduce the following properties:

PROPERTIES 1. (1) $K_n(\varnothing) = \varnothing$.

(2) $A \subset B \Rightarrow K_n(A) \subset K_n(B)$

(3) *For all non-increasing sequences $(A_k)_k$ of elements of $\mathscr{P}(E)$, there exists $k_0 \in \mathbb{N}$ such that*

$$K_n\Big(\bigcap_k A_k\Big) = K_n(A_{k_0}).$$

(4) *For all non-decreasing sequences $(A_k)_k$ of elements of $\mathscr{P}(E)$, there exists $k_1 \in \mathbb{N}$ such that*

$$K_n\Big(\bigcup_k A_k\Big) = K_n(A_{k_1}).$$

Proof. Properties 1 and 2 are trivial. Notice that the $(K_n(A_k))_k$ form a sequence of closed sets of D_n (compact), and that $K_n(\bigcap_k A_k) = \bigcap_k K_n(A_k)$ and $K_n(\bigcap_k A_k) = \bigcup_k K_n(A_k)$.

Property 3. Let us show that there exists a finite subset N_0 of $\mathbb{N}$ such that $K_n(\bigcap_k A_k) = \bigcap_{k \in N_0} K_n(A_k)$. Set $B = \bigcap_k K_n(A_k)$ and $B_p = K_n(A_p) \setminus B$. Then $(B_p)_p$ form a sequence of closed sets of D_n with empty intersection. We deduce that there exists a finite subset N_0 of $\mathbb{N}$ such that $\bigcap_{p \in N_0} B_p = \varnothing$, meaning that $\bigcap_{k \in N_0} K_n(A_k) \subset \bigcap_k K_n(A_k)$. This gives $\bigcap_k K_n(A_k) = \bigcap_{k \in N_0} K_n(A_k) = K_n(A_{k_0})$ with $k_0 = \max N_0$.

Property 4. Let us show that there exists a finite subset N_1 of $\mathbb{N}$ such that $K_n(\bigcup_k A_k) = \bigcup_{k \in N_1} K_n(A_k)$. The proof is similar to the previous one. One need only set $B = \bigcup_k K_n(A_k)$ and $B_p = B \setminus A_p$. This yields a sequence of closed sets of D_n, of empty intersection. From this sequence, we can extract a finite sequence of empty intersection; i.e., there exists a finite subset N_1 of $\mathbb{N}$ such that $\bigcap_{p \in N_1} B_p = \emptyset$, or equivalently $\bigcup_k K_n(A_k) \subset \bigcup_{k \in N_1} K_n(A_k)$, and thus $\bigcup_k K_n(A_k) = \bigcup_{k \in N_1} K_n(A_k) = K_n(A_{k_1})$ with $k_1 = \max N_1$. ∎

DEFINITION 3. We consider the set $\mathscr{O}$ of operators $\coprod$ from $\mathscr{P}(\mathbb{N}) \times (\mathbb{R}_+)^{\mathbb{N}}$ to $\mathbb{R}_+$ such that:

- The image of (I, u) by $\coprod$ is denoted $\coprod_{k \in I} u_k$.
- $\forall u \in (\mathbb{R}_+)^{\mathbb{N}}$, $\coprod_{\emptyset} u_k = 0$.
- for all $u \in (\mathbb{R}_+)^{\mathbb{N}}$ and $I_1 \subset I_2 \subset \mathbb{N}$, we have

$$\coprod_{k \in I_1} u_k \leq \coprod_{k \in I_2} u_k .$$

PROPOSITION 6. *Assume that the x_k^n are such that there exists a corresponding sequence of partitions $\mathscr{P}$ satisfying conditions* (C1) *and* (C2). *Let $\coprod$ be an $\mathscr{O}$ operator, and $(\zeta_n)_{n \geq 1}$ be a sequence of set functions, each ζ_n mapping $(I_k^n)_{0 \leq k < \nu_n}$ to $\mathbb{R}_+$. For all n and $A \in \mathscr{P}(E)$, we define*

$$c_n(A) := \coprod_{k \in K_n(A)} \zeta_n(I_k^n) .$$

Then, for every n, c_n is a Choquet $\mathscr{P}(E)$-capacity.

Proof. • If $A \subset B$, then $K_n(A) \subset K_n(B)$ and therefore $c_n(A) \leq c_n(B)$.

• Let $(A_k)_k$ be a non-decreasing sequence of subsets of E. The inequality $\sup_k c_n(A_k) \leq c_n(\bigcup_k A_k)$ is due to the monotony of c_n. Let us show the equality. There exists k_1 such that $K_n(\bigcup_k A_k) = K_n(A_{k_1})$, which gives $c_n(\bigcup_k A_k) = c_n(A_{k_1})$. This yields the desired result.

• Let (A_k) be a non-increasing sequence of elements of $\mathscr{P}(E)$. The inequality $\inf_k c_n(A_k) \geq c_n(\bigcap_k A_k)$ is due to the monotony of c_n. Let us show the equality. There exists k_0 such that $K_n(\bigcap_k A_k) = K_n(A_{k_0})$, which gives $c_n(\bigcap_k A_k) = c_n(A_{k_0})$. This yields the desired result. ∎

Remark. Let p, n be two integers, $p > n$, and $A \subset I_m^n$. Then

$$K_n(A) \subset K_n(I_m^n) = \{m\} .$$

Therefore, $K_n(A)$ is either $\emptyset$ or $\{m\}$, and thus

$$c_n(A) = 0 \qquad \text{or} \qquad c_n(A) = c_n(I_m^n) .$$

This means that, at a give "resolution" n, every set included in I_m^n is either not "seen" by c_n, or "measured" as I_m^n itself. For this reason, we shall call such capacities *myopic capacities*.

4.2. *Particular Cases*

The following definitions are useful in applicatons:

DEFINITION 4. We define the following applications from $\mathscr{P}(E)$ to $\overline{\mathbb{R}}_+$:

$$c_p^n(A) = \left(\sum_{k \in K_n(A)} \zeta_n(I_k^n)^p \right)^{1/p} \qquad \text{for } p \geq 1$$

$$c_\infty^n(A) = \max_{k \in K_n(A)} \zeta_n(I_k^n)$$

$$c_{-\infty}^n(A) = \min_{k \in K_n(A)} \zeta_n(I_k^n)$$

$$c_{\text{iso}}^n(A) = \max_t \operatorname{card}\{k \in K_n(A), \zeta_n(I_k^n) = t\}.$$

PROPOSITION 7. (1) *c_1^n is a measure.*

(2) *c_p^n is a Choquet capacity for $p > 1$.*

(3) *c_∞^n is a Choquet capacity.*

(4) *$c_{-\infty}^n$ is the inverse of a Choquet capacity, i.e., the set function $(c_{-\infty}^n)^{-1} := 1/c_{-\infty}^n$ is a Choquet capacity.*

(5) *c_{iso}^n is a Choquet capacity.*

Proof. (1) It is the particular case $\coprod = \sum$. The additivity of c_1^n is trivial, and yields the desired result (see the remark following the definition of capacities).

(2) Particular case $\coprod_{k \in I} u_k = (\sum_{k \in I} u_k^p)^{1/p}$.

(3) Particular case $\coprod = \max$.

(4) Particular case $\coprod = 1/\min$.

(5) Notice that

$$c_{\text{iso}}^n(A) = \max_t \operatorname{card} \operatorname{Ind}\big(K_n(A), t, \zeta\big)\}$$

with

$$\operatorname{Ind}(F, t, \zeta) = F \cap i_n^{-1}\big(X_n \cap \zeta_n^{-1}(t)\big)$$

(we identify $\zeta_n(I_k^n)$ and $\zeta_n(x_k^n)$) and that this application is non-decreasing with respect to F. We deduce the desired result by setting $\coprod_I u = \max_t \operatorname{card}\{\operatorname{Ind}(I, t, u)\}$. ∎

For a particular class of ζ_n, we have the results of convergence described in Proposition 8 (here again, we identify $\zeta_n(I_k^n)$ with $\zeta_n(x_k^n)$). First of all, we need recall a classical definition:

DEFINITION 5 [27]. Let $x_1, x_2, \ldots, x_n, \ldots$ be an infinite sequence such that $0 \leq x_n \leq 1$ for all n. Let $[\alpha, \beta]$ be an arbitrary subinterval of $[0, 1]$, and $N_n(\alpha, \beta)$ the number of x_i's, $i = 1, \ldots, n$, belonging to $[\alpha, \beta]$.

The sequence $x_1, x_2, \ldots, x_n, \ldots$ is called equidistributed if

$$\lim_{n \to +\infty} \frac{N_n(\alpha, \beta)}{n} = \beta - \alpha.$$

Roughly speaking, this definition means that the "probability" that a term x_n will fall into a certain subinterval of $[0, 1]$ is equal to the length of that subinterval, which is equivalent to saying that for every Riemann-integrable function over $[0, 1]$, the Riemann summation converges toward the Riemann integral [27].

PROPOSITION 8. *Let* $E := [0, 1[$, $A \in \mathscr{P}(E)$, $p \in \mathbb{N}$ *such that* $\zeta_n = \nu_n^{-1/p} f_n$ *with* $f_n \overset{\text{unif}}{\to} f$ *such that* f^p *is Riemann-integrable on* A. *Note that* $\|f\|_p = [\int_A (f(x))^p\, dx]^{1/p}$. *If the sequence* $(x_k^n)_{n \in \mathbb{N},\ 0 \le k < \nu_n}$ *is equidistributed, then we have the following results*:

(1) *If* $q > p$ *and* $\|f\|_q < +\infty$, *then* $c_q^n(A) \to 0$.

(2) *If* $q < p$ *and* $\|f\|_q \neq 0$, *then* $c_q^n(A) \to +\infty$.

(3) *If* $q = p$, $c_q^n(A) \to \|f\|_q$.

Proof.

$$\begin{aligned}
\left(c_q^n(A)\right)^q &= \sum_{k \in K_n(A)} \zeta_n(x_k^n)^q \\
&= \nu_n^{-q/p} \sum_{k \in K_n(A)} f_n(x_k^n)^q \\
&= \nu_n^{-(q/p-1)} \nu_n^{-1} \sum_{k \in K_n(A)} f_n(x_k^n)^q.
\end{aligned}$$

Set

$$\mu_n(A) = \nu_n^{-1} \sum_{k \in K_n(A)} f_n(x_k^n)^q \text{ and } \eta_n(A) = \nu_n^{-1} \Sigma_{k \in K_n(A)} f(x_k^n)^q.$$

Then

$$|\mu_n(A) - \eta_n(A)| \le \nu_n^{-1} \operatorname{card}(K_n(A)) \sup_{x \in E} |f_n^q(x) - f^q(x)|$$

and thus

$$\lim_n (\mu_n(A) - \eta_n(A)) = 0$$

(recall that $\operatorname{card}(K_n(A)) \le \nu_n$). Furthermore,

$$\eta_n(A) = \nu_n^{-1} \sum_{k=0}^{\nu_n - 1} (f \mathbb{1}_A)^q (x_k^n) \underset{n \to +\infty}{\to} \int_A f^p(x)\text{'}dx = \|f\|_q^q.$$

Therefore, $\lim_n \mu_n(A) = \|f\|_q^q$.

(1) If $q > p$, $\nu_n^{-(q/p-1)} \to 0$ and $\|f\|_q^q < +\infty$.

(2) If $q < p$, $\nu_n^{-(q/p-1)} \to +\infty$ and $\|f\|_q^q \neq 0$, which gives $(c_q^n(A))^q \to +\infty$.

(3) If $q = p$, $(c_q^n(A))^q = \mu_n(A) \to \|f\|_q^q$. ∎

5. CONSTRUCTION OF A SEQUENCE OF CAPACITIES WITH PRESCRIBED LIMIT SPECTRUM

In this section, we show how to construct a sequence of myopic capacities whose f_h spectrum is, under mild restrictions, prescribed.

We also prove that, given a measure μ whose f_h spectrum exists and satisfies some conditions, we can define from μ a sequence of myopic capacities whose f_h spectrum is any continuous monotonic function whose range is included in the range of the f_h spectrum of μ.

Finally, we show that, under some conditions, the f_l spectrum is the concave hull of the f_g spectrum, and in some cases, of the f_h spectrum.

As a warm-up, we first define a very particular sequence of capacities which is simple but contains most of the ideas we would like to address in this section.

5.1. *Iso-capacities*

Let ν be a probability measure on $[0,1[$, and $\mathscr{P} := ((I_k^n)_{0 \le k < \nu_n})_{n \ge 1}$ be a sequence of partitions satisfying (C1) and (C2).

Let $(l(n))_n$ be a sequence of strictly positive integers such that

$$\lim_{n \to +\infty} \frac{l(n)}{n + l(n)} = \beta$$

exists and belongs to $\mathbb{R}_+^*$.

For brevity, we shall drop the dependency upon n in the notations and simply write l. For any $A \subset [0,1[$, we define

$$c_n(A) = \max_{k \in K_n(A)} \zeta_n(I_k^n)$$

with

$$\zeta_{n+l}\left(I_k^{n+l}\right) = \frac{1}{2^l} \operatorname{card}\left\{j,\, \nu\left(I_j^{n+l}\right) = \nu\left(I_k^{n+l}\right) \text{ and } I_j^{n+l} \subset I_{a(k)}^n\right\},$$

where $I_{a(k)}^n$ is the unique interval which contains I_k^{n+l}.

We shall call $(c_n)_n$ the sequence of myopic iso-capacities associated with ν. Let us examine the case where ν is a binomial measure on $[0,1[$, with a dyadic partition, and the Lebesgue measure is the reference measure. For $t \in [0,1[$, we easily compute $c_{n+l}(I^{n+l}(t))$ to be

$$c_{n+l}\big(I^{n+l}(t)\big) = \frac{\binom{l}{S_{n,l}(t)}}{2^l},$$

where

$$S_{n,l}(t) = \sum_{i=n+1}^{n+l} (1-t_i)$$

with $t = \sum_{i=1}^{+\infty} t_i 2^{-i}$, $t_i = 0$ or 1 (dyadic expansion of t).

Let us denote

$$S_n(t) = \sum_{i=1}^{n} (1-t_i).$$

We have the following lemma:

LEMMA 2. *Assuming $\beta > 0$ exists, then the sequence $(S_{n,l}(t)/l)_n$ converges if and only if the sequence $(S_n(t)/n)_n$ converges and, in this case, they have the same limit.*

Proof. Set

$$u_n(t) := \frac{S_n(t)}{n}, \qquad v_{n,l}(t) := \frac{S_{n,l}(t)}{l}.$$

From the definition of S_n, we have

$$\begin{aligned} u_{n+l}(t) &= \frac{1}{n+l}\sum_{i=1}^{n+l}(1-t_i) \\ &= \frac{n}{n+l}u_n(t) + \frac{l}{n+l}v_{n,l}(t). \end{aligned}$$

Hence,

$$v_{n,l}(t) = \frac{u_{n+l}(t) - (1 - l/(n+l))u_n(t)}{l/(n+l)}.$$

- If $u_n(t) \to u(t)$, then clearly $v_{n,l}(t) \to u(t)$.

- Assume that $v_{n,l}(t)$ converges. From $u_n(t)$, we can extract a sequence $(u_{\sigma(n)}(t))_n$ such that $u_{\sigma(n)}(t) \to \bar{u}(t)$, and hence $u_{\sigma(n)+l(\sigma(n))}(t) \to \bar{u}(t)$, which yields

$$v_{\sigma(n),l(\sigma(n))}(t) \to \frac{\bar{u}(t) - (1-\beta)\bar{u}(t)}{\beta} = \bar{u}(t)$$

and therefore

$$\lim_{n\to+\infty} v_{n,l}(t) = \bar{u}(t).$$

This shows that all the convergent subsequences of $u_n(t)$ have the same limit (namely $\lim_{n\to+\infty} v_{n,l(n)}(t)$). Therefore, $u_n(t)$ converges toward $\lim_{n\to+\infty} v_{n,l(n)}(t)$. ∎

Let $t \in [0,1[$ be such that

$$\varphi_0(t) = \lim_{n\to+\infty} \frac{S_n(t)}{n}$$

exists. Then a straightforward computation gives

$$\alpha_{\text{iso}}(t) = \lim_{n\to+\infty} \frac{\log c_n(I^n(t))}{\log|I^n(t)|} = \beta\big(1 - f(\varphi_0(t))\big)$$

with $f(u) = -u\log_2 u - (1-u)\log_2(1-u)$.

On the other hand, for $\alpha \in [0, \beta]$,

$$E_{\alpha_{\text{iso}}} = \{t,\ \alpha_{\text{iso}}(t) = \alpha\} = \{t,\ \varphi_0(t) = \varphi_0\} \cup \{t \,|\, \varphi_0(t) = 1 - \varphi_0\},$$

where φ_0 is one of the two solutions of

$$\alpha = \beta\big(1 - f(\varphi_0)\big).$$

The two sets in the union above have the same Hausdorff dimension, namely $f(\varphi_0)$. Thus we obtain the following:

PROPOSITION 9. *The f_h spectrum of the sequence $(c_n)_n$ is given by*

$$\alpha \in [0, \beta]$$

$$f_h(\alpha) = 1 - \frac{\alpha}{\beta}.$$

Note that α is $\mathscr{L}$-a.s. equal to 0. More precisely, if t is 2-normal, then $\lim_{n\to+\infty} c_n(t) = 1$ and $\alpha_{\text{iso}}(t) = 0$. Otherwise, $\lim_{n\to+\infty} c_n(t) = 0$. Also, for any interval $I \subset [0,1[$, with $|I| > 0$, $\lim_{n\to+\infty} c_n(I) = 1$.

Here, we have constructed a sequence of capacities whose f_h spectrum is a line segment. We generalize this result in the following section.

5.2. *Controlling the Shape of the f_h Spectrum*

The iso-capacities introduced in Section 5.1 prove a special case of the myopic capacities to be defined in the proof of the Theorem 2. First of all, we need some definitions.

Let $\mathscr{C}$ be the space of all Choquet capacities defined on $[0, 1[$, and taking values in $[0, 1]$.

Let $\mathscr{F}$ be the space of all functions from $\mathbb{R}^+$ to $[0, 1] \cup \{-\infty\}$.

Define

$$D(f) := \{\alpha \in \mathbb{R}^+ \,|\, f(\alpha) \neq -\infty\}.$$

For a *closed* subset A of $\mathbb{R}^+$, define

$$\begin{aligned} \mathscr{F}(A) := \{ & f \in \mathscr{F}, D(f) = A \text{ and } f \text{ is either invertible with} \\ & f^{-1} \text{ continuous, or } f \text{ is identically zero on } A\} \\ \mathscr{F}^0 &:= \bigcup_{A \text{ closed}} \mathscr{F}(A) \\ \mathscr{F}^1 &:= \{f \in \mathscr{F}, \exists (f_n)_{n \geq 1}, f_n \in \mathscr{F}^0 \text{ for all } n \text{ and } f = \sup_n f_n\}. \end{aligned}$$

On the other hand, let us denote by $\mathscr{S}$ the set of all functions that are the f_h spectra of a sequence of capacities belonging to $\mathscr{C}$, i.e.,

$$f \in \mathscr{S} \Leftrightarrow \exists c := (c_n)_{n \geq 1}, \forall n, c_n \in C \text{ and } f_{h,c} = f.$$

The following theorem shows that every element of $\mathscr{F}^1$ is the f_h spectrum of a sequence of $\mathscr{C}$.

THEOREM 2. $\mathscr{F}^1 \subset \mathscr{S}$.

Throughout the rest of the paper, we shall say that μ is a reference measure if and only if μ is a non-atomic probability measure defined on $[0, 1[$.

We prove the theorem in several steps.

DEFINITION 6. Let μ be a reference measure, and ν be a probability measure defined on $[0, 1[$. ν is said to verify property $(\mathscr{P}_\mu)$ if and only if there exists a non-empty *closed* subset Δ of $D(f_{h,\nu})$ such that the restriction of $f_{h,\nu}$ to Δ is continuous and invertible. ν is said to verify property $(\mathscr{P}'_\mu)$ if and only if there exists a non-empty *open* subset Δ of $D(f_{h,\nu})$ such that the restriction of $f_{h,\nu}$ to Δ is continuous and invertible.

We first note that there indeed exist reference measures μ such that the set of probability measures ν verifying $(\mathscr{P}_\mu)$ or $(\mathscr{P}'_\mu)$ is not empty. Take, for instance μ to be the Lebesgue measure, ν to be a binomial measure with weights (m_0, m_1), $m_0 < m_1$, and

$$\Delta := \left[-\log_2 m_1, -\frac{\log_2 m_0 m_1}{2} \right] \quad \text{for } (\mathscr{P}_\mu) \qquad \text{and}$$

$$\Delta := \left] -\log_2 m_1, -\frac{\log_2 m_0 m_1}{2} \right[\text{ for } (\mathscr{P}'_\mu).$$

The key result for proving the theorem is

PROPOSITION 10. *Let μ be a reference measure such that there exists a probability measure ν satisfying $(\mathscr{P}_\mu)$. Let Δ be a closed subset of $D(f_{h,\nu})$ such that the restriction of $f_{h,\nu}$ to Δ is continuous and inveritible.*

Let A and B be two closed sets, $A \subset \mathbb{R}^+$, $B \subset f_{h,\nu}(\Delta)$. Let s be an invertible function from A onto B such that $s^{-1} \circ f_{h,\nu}|_F$ is continuous, where $F := f_{h,\nu}^{-1}(B)$. Then there exists a sequence $c := (c_n)_{n \geq 1}$ of $\mathscr{C}$ such that

$$f_{h,c}|_A = s$$

$$f_{h,c}|_{A^c} = -\infty,$$

where $A^c := \mathbb{R}^+ \setminus A$.

Proof. Let $\mathscr{P} := ((I_k^n)_k)_{n \geq 1}$ be the analyzing sequence of partitions verifying (C1) and (C2), used for the computation of $f_{h,\nu}$. Notice that the assumption $B \subset f_{h,\nu}(\Delta)$ implies $F \neq \emptyset$, and F is closed since $f_{h,\nu}|_\Delta$ is continuous.

Recall that

$$\alpha_n(x) := \frac{\log \nu(I^n(x))}{\log \mu(I^n(x))}.$$

As in Section 4, we associate to $\mathscr{P}$ a sequence $(X_n)_{n \geq 1}$ such that, for every n, $X_n := \{x_k^n;\ 0 \leq k < \nu_n\}$, and each I_k^n contains exactly one x_k^n.

Set

$$G_n := \left\{ \alpha_n(x_k^n);\ k = 0, \dots, \nu_n - 1 \right\}$$

$$G := \bigcup_{n \geq 1} G_n.$$

Let $x \in [0,1[$ be such that $\alpha(x)$ exists. Then

$$\alpha(x) := \lim_n \alpha_n(x) = \lim_n \alpha_n\left(x_{k_n(x)}^n\right),$$

where $k_n(x)$ is the (unique) integer such that $I_{k_n(x)}^n = I^n(x)$. Hence $\alpha(x) \in G$, which gives

$$F \subset D(f_{h,\nu}) \subset G.$$

Define

$$f := f_{h,\nu}|_F$$
$$g := s^{-1} \circ f.$$

g is a continuous one-to-one function from F onto A. Since F and G are closed, there exists, by Tietze's extension theorem, a continuous function $\tilde{g}$ defined on G such that

$$\tilde{g}|_F = g.$$

We can now construct the desired sequence $c = (c_n)_{n \in \mathbb{N}}$ of Choquet capacities. The c_n will be myopic capacities as defined in Section 4, and are given by (using the notation of Section 4)

$$\zeta_n(I_k^n) := \mu(I_k^n)^{\tilde{g}(\alpha_n(x_k^n)) + u_n d(\alpha_n(x_k^n), F)}$$
$$c_n(A) := \max_{k \in K_n(A)} \zeta_n(I_k^n) \qquad \text{for } A \subset [0,1[,$$

where $(u_n)_{n \in \mathbb{N}}$ is a bounded and non-convergent sequence of non-negative real numbers (e.g., $u_n = 1 + (-1)^n$), and $d(x, F)$ denotes the distance between x and F.

Set

$$\beta_n(x) := \frac{\log c_n(I^n(x))}{\log \mu(I^n(x))}.$$

This gives

$$\beta_n(x) = \tilde{g}(\alpha_n(x)) + u_n d(\alpha_n(x), F).$$

We are led to distinguish two situations:

(1) $\alpha(x) \in F$. In this case, using the continuity of $d(\cdot, F)$ and $\tilde{g}$,

$$\beta(x) := \lim_n \beta_n(x) = \tilde{g}(\alpha(x)) = s^{-1} \circ f(\alpha(x)).$$

(2) $\alpha(x) \notin F$. Then $d(\alpha(x), F) > 0$, and since $\{u_n\}$ does not converge, neither does $\beta_n(x)$.

Finally,

$$\alpha(x) \in F \Rightarrow \beta(x) \text{ exists}$$
$$\alpha(x) \notin F \Rightarrow \beta(x) \text{ does not exist}.$$

We still need to prove that if $\alpha(x)$ does not exist, neither does $\beta(x)$.

Assume that $\alpha(x)$ does not exist Then, from $(\alpha_n(x))_n$, we can extract two sequences, $(\alpha_{n_1}(x))_{n_1}$ and $(\alpha_{n_2}(x))_{n_2}$, converging toward different values $\alpha_1(x)$ and $\alpha_2(x)$, respectively.

We consider the cases:

(1) $\alpha_1(x) \in F$ and $\alpha_2(x) \in F$. In this case, $\beta_{n_1}(x) \to s^{-1} \circ f(\alpha_1(x))$ and $\beta_{n_2}(x) \to s^{-1} \circ f(\alpha_2(x))$. Since the two limits are different (because $s^{-1} \circ f$ is invertible), $\beta(x)$ does not exist.

(2) $\alpha_1(x) \in F$ and $\alpha_2(x) \notin F$. In this case, $\beta_{n_1}(x) \to s^{-1} \circ f(\alpha_1(x))$, and $\beta_{n_2}(x)$ does not converge. Thus $\beta(x)$ does not exist.

(3) $\alpha_1(x) \notin F$ and $\alpha_2(x) \notin F$. In this case, neither $\beta_{n_1}(x)$ nor $\beta_{n_2}(x)$ converges, and again, $\beta(x)$ does not exist. Finally,

$$\alpha(x) \text{ does not exist} \Rightarrow \beta(x) \text{ does not exist}.$$

Set

$$D := \{x \in [0,1[,\ \alpha(x) \text{ exists}\}$$
$$X := \{x \in D,\ \alpha(x) \in F\}.$$

We just proved that

$$x \in X \Leftrightarrow \beta(x) \text{ exists}.$$

Let $\beta \in A$. Then

$$\begin{aligned} E_\beta &:= \{x \in X,\ \beta(x) = \beta\} \\ &= \{x \in X,\ \alpha(x) = f^{-1} \circ s(\beta)\} \\ &= \{x \in D,\ \alpha(x) = f^{-1} \circ s(\beta)\} \qquad (\text{since } f^{-1} \circ s \text{ maps } A \text{ onto } F). \end{aligned}$$

Hence

$$f_{h,c}(\beta) := \dim_\mu E_\beta = f_{h,\nu}(f^{-1} \circ s(\beta)) = s(\beta).$$

It is straightforward to check that $E_{\beta,c} = \emptyset$ if $\beta \notin A$. ∎

The previous proposition shows that the elements of $\mathscr{F}^0$ which are not identically zero belong to $\mathscr{S}$ (in fact, it proves a little more, since we only need $s^{-1} \circ f$ to be continuous and not necessarily s^{-1}).

The following proposition takes care of the case $f = 0$ on $D(f)$.

PROPOSITION 11. *Let F be a closed subset of $\mathbb{R}^+$. There exists a sequence $c^0 := (c_n^0)_{n\geq 1}$ of $\mathscr{C}$ such that*

$$f_{h,c^0}(\alpha) = \begin{cases} 0 & \text{if } \alpha \in F \\ -\infty & \text{if } \alpha \notin F. \end{cases}$$

Proof. Let $\mathscr{P} := ((I_k^n)_k)_n$ be a sequence of partitions verifying conditions (C1) and (C2), and consider the associated sequence $((x_k^n)_k)_n$ as in the proof of Proposition 10. Let $(u_n)_n$ be a non-convergent and bounded sequence of $\mathbb{R}^+$.

Define

$$\zeta_n(I_k^n) := \mu(I_k^n)^{x_k^n + u_n d(x_k^n, F)}.$$

μ is any reference measure, and $c_n^0(U) := \max\{\zeta_n(I_k^n) \mid k \in K_n(U)\}$ for all $U \subset \mathbb{R}^+$. For $x \in [0,1[$, let $k_n(x)$ be the index of the (unique) interval I_k^n containing x. Clearly, $x_{k_n(x)}^n \to x$ when n tends to infinity. This gives

$$\beta_n(x) := \frac{\log c_n^0(I^n(x))}{\log \mu(I^n(x))} = x_{k_n(x)}^n + u_n d\big(x_{k_n(x)}^n, F\big)$$

and $\beta(x) := \lim_n \beta_n(x) = x$ if $x \in F$. If $x \notin F$, then $\beta_n(x)$ does not converge. The f_h spectrum is given by

$$f_{h,c^0}(\beta) := \dim_\mu\{x \in [0,1[,\, \beta(x) = \beta\} = \begin{cases} 0 & \text{if } \beta \in F \\ -\infty & \text{if } \beta \notin F. \end{cases} \quad \blacksquare$$

Proposition 10 together with Proposition 11 proves that $\mathscr{F}^0 \subset \mathscr{S}$. To prove $\mathscr{F}^1 \subset \mathscr{S}$, we use the following construction:

Proof of Theorem 2. Let $f \in \mathscr{F}^1$. Then there exists a sequence $(f_i)_{i\geq 1}$ of $\mathscr{F}^0$ such that $f = \sup_i f_i$. Let $A_i := D(f_i)$, and $A := \bigcup_i A_i$. A is an F_σ set (since every A_i is closed). Using Propositions 10 and 11, we may find, for every i, an element of $\mathscr{C}$, say $c_i := (c_n^{(i)})_{n\geq 1}$, such that $f_{h,c_i} = f_i$. Without loss of generality, we may assume that every $c_n^{(i)}$ is null outside $I_i := [1/(i+1), 1/i[$. Set, for every n, $c_n := \sup_{i\geq 1} c_n^{(i)}$ and $c := (c_n)_{n\geq 1}$. Then $c \in \mathscr{C}$. Since the I_i's are disjoint, one easily shows that

$$E_{\alpha,c} := \bigcup_i E_{\alpha,c_i}$$

and thus

$$f_{h,c} = \sup_{i\geq 1} f_{h,c_i} = \sup_{i\geq 1} f_i = f. \quad \blacksquare$$

EXAMPLE 1. Let F be an F_σ subset of $\mathbb{R}^+$. There exists a sequence $c^z := (c_n^z)_n$ of $\mathscr{C}$ such that

$$f_{h,c^z}(\alpha) = \begin{cases} 0 & \textit{if } \alpha \in F \\ -\infty & \textit{if } \alpha \notin F. \end{cases}$$

Proof. Let $F := \bigcup_i F_i$, where F_i is closed in $\mathbb{R}^+$. The sequence $(F_i)_i$ may be chosen such that $F_i \subset F_{i+1}$.

From Proposition 11, we know that for every i, there exists a sequence $c^{(i)}$ of $\mathscr{C}$ such that

$$f_{h,c^{(i)}}(\alpha) = \begin{cases} 0 & \text{if } \alpha \in F_i \\ -\infty & \text{if } \alpha \notin F_i. \end{cases}$$

Therefore, for all i, $f_{h,c^{(i)}} \in \mathscr{F}^0$ and, by Theorem 2, there exists $c_z \in \mathscr{C}$ such that

$$f_{h,c^z} = \sup_i f_{h,c^{(i)}}.$$

Since $F_i \subset F_{i+1}$, one easily verifies that

$$f_{h,c^z}(\alpha) = \begin{cases} 0 & \text{if } \alpha \in F \\ -\infty & \text{if } \alpha \notin F. \end{cases} \quad \blacksquare$$

Remark. In the following, we shall refer to f^z for f_{h,c^z} when considering the particular case $F = \mathbb{R}^+$, and we keep the notation c^z for the corresponding myopic capacity. Thus, $f^z = 0$ on $\mathbb{R}^+$. Note that the support of c^z can be chosen arbitrarily among subintervals of $[0,1[$, semi-open to the right.

EXAMPLE 2. Let F be an F_σ subset of $\mathbb{R}^+$,and $\alpha \in]0,1]$. Then

$$\alpha \mathbb{1}_F \in S.$$

Proof. Suppose first that F is closed. For all p, let f_p be the restriction to F of

$$x \mapsto \alpha \left(\frac{x+1}{x+2} \right)^{1/p}.$$

Clearly, $f_p \in \mathscr{F}^0$ for all p, and by Theorem 2,

$$\sup_p f_p \in \mathscr{F}^1.$$

Thus, there exists a sequence $c := (c_n)_n$ of $\mathscr{C}$ such that

$$f_{h,c} = \begin{cases} \alpha & \text{on } F \\ -\infty & \text{elsewhere.} \end{cases}$$

We can choose (see the remark above) a sequence c null outside $[0, \frac{1}{2}[$.

If we consider the sequence c^z of Example 1 (we can choose a sequence c^z null outside $[\frac{1}{2}, 1[$), then setting

$$d := \max(c, c^z)$$

gives

$$f_{h,d} = \alpha \, \mathbb{1}_F .$$

The generalization to F_σ sets can be easily deduced by following the lines of the proof of Example 1. ∎

Since both Cantor sets and the set of all positive rationals are F_σ sets, we have

EXAMPLE 3. $\mathbb{1}_{\text{Cantor}} \in \mathscr{S}$ and $\mathbb{1}_{Q^+} \in \mathscr{S}$.

The following proposition is a weak version of Proposition 10, but is more appropriate for practical purposes as it permits us to construct the desired sequence of capacities very easily.

PROPOSITION 12. *Let $\mathscr{P} := ((I_k^n)_k)_n$ be a sequence of partitions verifying conditions $(C1)$ and $(C2)$. Let μ be a reference measure, and ν a probability measure verifying $(\mathscr{P}'_\mu)$. Let $\Delta \subset D(f_{h,\nu})$ be an open set such that $f_{h,\underline{\nu}}$ is continuous and invertible on Δ. Let A, be two open sets, $A \subset \mathbb{R}^+$, $\overline{B} \subset f_{h,\nu}(\Delta)$, such that $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$, where $f := f_{h,\nu}|_\Delta$. Let s be an invertible function from $\overline{A}$ onto $\overline{B}$ such that $s^{-1} \circ f$ is continuous and $s(A) = B$. Then there exists a sequence $c = (c_n)_{n \geq 1}$ of $\mathscr{C}$ such that*

$$f_{h,c}|_A = s|_A$$

$$f_{h,c}|_{\partial A} \leq s|_{\partial A}$$

$$f_{h,c}|_{(\overline{A})^c} = -\infty,$$

where $f_{h,c}$ is the f_h spectrum of c with respect to the measure μ.

Set $\mathscr{O} := f^{-1}(B)$, $D := \{x \in [0 \,|\, 1[, \ \alpha(x) \text{ exists}\}$, and

$$L := \{x \in D, \alpha(x) \in \partial O\}, \qquad R := \underline{\lim}_n \{x \in D, \alpha_n(x) \in O\}.$$

If $L \setminus R = \emptyset$ or $\dim_\mu(L \setminus R) = 0$, then

$$f_{h,c}|_{\partial A} = s|_{\partial A}.$$

Proof. Recall that, for all $x \in [0,1[$,

$$\alpha_n(x) := \frac{\log c_n(I^n(x))}{\log \mu(I^n(x))}$$

and $\alpha(x) := \lim_n \alpha_n(x)$ when this limit exists. Set $R_n := \{x \in D;\ \alpha_n(x) \in O\}$ and $X := \{x \in D;\ \alpha(x) \in O\}$ (hence $R = \underline{\lim}_n R_n$). The desired (myopic) capacity is constructed using $\coprod = \max$ and

$$\zeta_n(I_k^n) := \begin{cases} \mu(I_k^n)^{s^{-1} \circ f(\alpha_n(x_k^n))} & \text{if } \alpha_n(x_k^n) \in O \\ 0 & \text{otherwise}. \end{cases}$$

Following the lines of the proof of Proposition 10, it is clear that, if $x \in X \cup (R \cap L)$, $\beta(x)$ exists and equals $s^{-1} \circ f(\alpha(x))$. Otherwise, either $\beta_n(x)$ does not exist, or it does not converge and thus $\beta(x)$ is not defined. This implies in particular that, if $\beta \notin \overline{A}$, then $E_{\beta,c} = \emptyset$.

Notice that

$$\alpha \in O \Leftrightarrow s^{-1} \circ f(\alpha) \in A.$$

We will need the following lemma:

LEMMA 3. $\beta \in \partial A \Leftrightarrow f^{-1} \circ s(\beta) \in \partial O$.

Proof. Let us prove $\beta \in \partial A \Rightarrow f^{-1} \circ s(\beta) \in \partial O$. We have

$$\beta \in \overline{A} \Rightarrow f^{-1} \circ s(\beta) \in f^{-1} \circ s(\overline{A}) = f^{-1}(\overline{B}) = \overline{f^{-1}(B)} = \overline{O}$$

and

$$f^{-1} \circ s(\beta) \in O \Rightarrow \beta \in s^{-1} \circ f(O) = A.$$

Conversely,

$$f^{-1} \circ s(\beta) \in \overline{O} \Rightarrow \beta \in s^{-1} \circ f(\overline{B}) = \overline{A}$$

$$\beta \in A \Rightarrow f^{-1} \circ s(\beta) \in f^{-1} \circ s(A) = O. \qquad \blacksquare$$

(1) If $\beta \in A$, then

$$\begin{aligned} E_{\beta,c} &:= \{x \in X \cup (R \cap L), \beta(x) = \beta\} \\ &= \{x \in X \cup (R \cap L), \alpha(x) = f^{-1} \circ s(\beta)\}. \end{aligned}$$

Since $f^{-1} \circ s$ maps A onto O, we have

$$E_{\beta,c} = \{x \in X, \alpha(x) = f^{-1} \circ s(\beta)\} = E_{f^{-1} \circ s(\beta),\nu},$$

which gives

$$f_{h,c}|_A = s|_A.$$

(2) If $\beta \notin \overline{A}$, then $E_{\beta,c} = \emptyset$ and

$$f_{h,c}|_{\overline{A}^c} = -\infty.$$

(3) If $\beta \in \partial A$, then, since $f^{-1} \circ s$ maps ∂A onto ∂O (Lemma 3),

$$\begin{aligned} E_{\beta,c} &:= \{x \in X \cup (R \cap L), \alpha(x) = f^{-1} \circ s(\beta)\} \\ &= \{x \in R \cup L, \alpha(x) = f^{-1} \circ s(\beta)\} \\ &\subset \{x \in L, \alpha(x) = f^{-1} \circ s(\beta)\} =: E_{f^{-1} \circ s(\beta), \nu}. \end{aligned}$$

Hence,

$$f_{h,c}|_{\partial A} \le s|_{\partial A}$$

Notice that

$$E_{f^{-1} \circ s(\beta), \mu} = E_{\beta c} \cup \{x \in L \setminus R, \alpha(x) = f^{-1} \circ s(\beta)\},$$

which yields

$$s(\beta) = \max\left(f_{h,c}(\beta), \dim_\mu\{x \in L \setminus R, \alpha(x) = f^{-1} \circ s(\beta)\}\right).$$

Thus, if $\dim_\mu(L \setminus R) = -\infty$ or 0, then

$$f_{h,c}|_{\partial A} = s|_{\partial A}. \quad \blacksquare$$

The following two results allow a better knowledge of f_h and $\tilde{f}_g$ for a certain class of capacities.

THEOREM 3. *Let* $\mathscr{P} := ((I_k^n)_{0 \le k < \nu_n})_{n \in \mathbb{N}}$ *be a sequence of partitions verifying conditions* (C1) *and* (C2), *and* $c := (c_n)_n$ *be a sequence of Choquet capacities. Let* μ *be a reference measure, and* $(\lambda_n)_{n \ge 1}$ *be a sequence of integers verifying* (1), *Section* 2.1.4. *Set*

$$G_n := \left\{ \frac{\log c_n(I_k^n)}{\log \mu(I_k^n)} \,\middle|\, 0 \le k < \nu_n \right\}.$$

If

$$\lim_n \frac{\log \operatorname{card} G_n}{\lambda_n} = 0$$

then

$$f_{l,c} = \tilde{f}^{**}_{g,c}.$$

Proof. Since $\tilde{f}_g \le f_l$, we already have $\tilde{f}_g^{**} \le f_l$. We shall only prove the opposite inequality. Set

$$\alpha_n(x_k^n) := \frac{\log c_n(I_k^n)}{\log \mu(I_k^n)}.$$

By assumption, we have

$$G_n := \{\alpha_n(x_k^n), 0 \le k < \nu_n\} =: \bigcup_{i=1}^{N(n)} \{\alpha_{i,n}\}$$

with

$$\lim_n \frac{\log N(n)}{\lambda_n} = 0.$$

Let

$$K_{i,n} := \{0 \le k < \nu_n, \alpha_n(x_k^n) = \alpha_{i,n}\}.$$

Replacing $c_n(I_k^n)$ by $\mu(I_k^n)^{\alpha_n(x_k^n)}$ gives, for all q and τ,

$$X_n(q-1,\tau) := \sum_{k<\nu_n} \mu(I_k^n)^{q\alpha_n(x_k^n)-\tau} = \sum_{i=1}^{N(n)} \sum_{k \in K_{i,n}} \mu(I_k^n)^{q\alpha_{i,n}-\tau}.$$

Set

$$i(n) := \operatorname*{argmax}_{i=1,\dots,N(n)} \sum_{k \in K_{i,n}} \mu(I_k^n)^{q\alpha_{i,n}-\tau}.$$

($i(n)$ may be not unique. In that case, take the smallest one, for instance.) This yields (we denote K_n for $K_{i(n),n}$ and α_n for $\alpha_{i(n),n}$)

$$\forall n \qquad X_n(q-1,\tau) \le N(n) \sum_{k \in K_n} \mu(I_k^n)^{q\alpha_n-\tau}.$$

This inequality holds in particular for those indices n' of the subsequence of $\log X_n(q-1,\tau)/\lambda_n$ converging toward $X(q-1,\tau)$.

Furthermore, we can extract from $(\alpha_{n'})$ a sequence (α_p) converging toward a limit denoted α. Recall that, in the definition of $\tilde{f}_g$, we have, for all $\varepsilon > 0$,

$$K_p^\varepsilon(\alpha) := \{k < \nu_p, \alpha_p(x_k^p) \in [\alpha-\varepsilon, \alpha+\varepsilon]\}.$$

When $k \in K_p$, then $g_p(x_k^p) = \alpha_{i(p), p} =: \alpha_p$. For all $\varepsilon > 0$, there exists p_0 such that for all $p \geq p_0$,

$$\alpha_p \in [\alpha - \varepsilon, \alpha + \varepsilon].$$

Hence,

$$\forall \varepsilon > 0, \exists p_0, \forall p \geq p_0 \qquad K_p \subset K_p^\varepsilon(\alpha)$$

and

$$X_p(q-1, \tau) \leq N(p) \sum_{k \in K_p^\varepsilon(\alpha)} \mu(I_k^p)^{q\alpha_p - \tau} \leq N(p) \sum_{k \in K_p^\varepsilon(\alpha)} \mu(I_k^p)^{q(\alpha \pm \varepsilon) - \tau}$$

$$= N(p) S_\varepsilon^p(\alpha, q(\alpha \pm \varepsilon) - \tau)$$

(the choice between $+$ or $-$ depends upon the sign of q).

Let τ be such that $\tilde{f}_g^\varepsilon(\alpha) > q(\alpha \pm \varepsilon) - \tau$. Then $S_\varepsilon(\alpha, q(\alpha \pm \varepsilon) - \tau) = 0$, and hence there exists $c > 0$ such that, for all large p,

$$S_\varepsilon^p(\alpha, q(\alpha \pm \varepsilon) - \tau) < c,$$

which yields

$$X(q-1, \tau) \leq \lim_p \frac{\log(cN(p))}{\gamma_p} = 0.$$

This implies $\tau \leq \tau(q)$. We thus proved

$$\forall q, \exists \alpha \text{ such that } \forall \varepsilon > 0, \qquad q(\alpha \pm \varepsilon) - \tilde{f}_g^\varepsilon(\alpha) \leq \tau(q).$$

When ε decreases to 0, we obtain

$$\tau(q) \geq q\alpha - \tilde{f}_g(\alpha) \geq \tilde{f}_g^*(q).$$

We conclude that

$$\tilde{f}_g^{**} \geq \tau^* = f_l. \quad \blacksquare$$

The condition $\lim_n((\log \operatorname{card} G_n)/\lambda_n) = 0$ is not necessary. Indeed, the following example gives a sequence $c := (c_n)_{n \geq 1}$ of $\mathscr{C}$ which does not verify this condition, and yet is such that $f_{l,c} = \tilde{f}_g^{**}$.

EXAMPLE 4. Choose $I_k^n := [k2^{-n}, (k+1)2^{-n}[$, $\nu_n = 2^n$, $\lambda_n = n$, and $\zeta(I_k^n) = 2^{-nk2^{-n}}$ and c_n the associated myopic capacity with $\coprod = \max$.

Computation of f_h. Let $x \in [0,1[$. We have

$$\alpha_n(x) := \frac{\log c_n(I^n(x))}{\log \mu(I^n(x))} = k_n(x) 2^{-n} \to x \qquad \text{when } n \to +\infty,$$

where $k_n(x) := [2^n x]$ (i.e., $I^n(x) = I^n_{k_n(x)}$). Thus, for all $\alpha \in [0,1[$,

$$E_\alpha := \left\{ x \in [0,1[\,\middle|\, \lim_n \alpha_n(x) = \alpha \right\} = \{\alpha\}$$

and $E_\alpha = \emptyset$ for $\alpha \in [1,+\infty[$. We obtain

$$f_h(\alpha) = \begin{cases} 0 & \text{if } \alpha \in [0,1[\\ -\infty & \text{if } \alpha \in [1,+\infty[. \end{cases}$$

Computation of f_g. Let $\alpha \in \mathbb{R}^+$, $n \in \mathbb{N}$, and $\varepsilon \in]0,1[$. Then

$$\begin{aligned} N_\varepsilon^n(\alpha) &:= \left\{ 0 \le k < 2^n \,\middle|\, \frac{\log c_n(I_k^n)}{\log \mu(I_k^n)} \in [\alpha - \varepsilon, \alpha + \varepsilon] \right\} \\ &= \left\{ 0 \le k < 2^n \middle| 2^n(\alpha - \varepsilon) \le k \le 2^n(\alpha + \varepsilon) \right\}. \end{aligned}$$

- If $\alpha = 0$, then

$$N_\varepsilon^n(0) = \left[2^n(\alpha + \varepsilon)\right] + 1,$$

which gives $f_g(0) = 1$.

- If $\alpha = 1$, then

$$N_\varepsilon^n(1) = 2^n - \left[2^n(\alpha + \varepsilon)\right],$$

which gives $f_g(1) = 1$.

- If $\alpha \in]0,1[$, then

$$N_\varepsilon^n(1) = \left[2^n(\alpha + \varepsilon)\right] - \left[2^n(\alpha - \varepsilon)\right] + 1,$$

which gives $f_g(\alpha) = 1$.

- If $\alpha > 1$, then for all $\varepsilon \in]0, \alpha - 1[$, we have $N_\varepsilon^n(\alpha) = 0$, and thus $f_g(\alpha) = -\infty$.

Computation of f_l. From Proposition 3, we deduce $f_l = f_g$.

PROPOSITION 13. *Let* $(c_i)_{i=1,\dots,p}$ *be p sequences* $\mathscr{C}$ $(c_i := (c_{n,i})_n)$, *taking values in* $[0;1[$, *and let* μ *be a reference measure. Assume that, for all* i, n, $c_{n,i}$ *is null outside* $[a_i; b_i[$, *where* $a_i < b_i \le a_{i+1}$, *and* $\bigcup_{i=1}^p [a_i, b_i[= [0,1[$.

Let $((I_k^{n,i})_k)_n$ *be a sequence of partitions of* $[a_i; b_i[$ *in intervals verifying conditions* (C1) *and* (C2). *Define*

$$\forall n \qquad c_n = \frac{1}{p}\sum_{i=1}^{p} c_{n,i}$$

and $c := (c_n)_n$. *Thus* $((I_k^{n,i})_i)_{k,n}$ *is a sequence of partitions of* $[0; 1[$ *satisfying conditions* (C1) *and* (C2). *If*

$$f_{l,c_i} = \tilde{f}^{**}_{g,c_i}$$

then

$$f_{l,c} = \tilde{f}^{**}_{g,c}.$$

If

$$f_{l,c_i} = f^{**}_{h,c_i}$$

then

$$f_{l,c} = f^{**}_{h,c}.$$

For simplicity, we shall write $f_{l,i}$ for f_{l,c_i}, $f_{g,i}$ for f_{g,c_i}, and $f_{h,i}$ for f_{h,c_i}. For a sequence $c := (c_n)_{n \geq 1}$ of $\mathscr{C}$, let τ_c denote the τ function (see Section 2.1.4) associated with c.

We will need the following lemma:

LEMMA 4. *Let* $(c_i)_{i=1,\ldots,p}$, c *and* μ *be the quantities defined in Proposition* 13, *and consider the partition defined therein*. *Then*

$$\tau_c = \min_{i=1,\ldots,p} \tau_{c_i}.$$

Proof. Denote τ_i for τ_{c_i}. For all q, τ,

$$\begin{aligned} X_n(q-1,\tau) &= \sum_k c_n(I_k^n)^q \mu(I_k^n)^{-\tau} \\ &= \sum_{i=1}^{p} \left(\sum_k c_{n,i}\left(I_k^{n,i}\right)^q \mu\left(I_k^{n,i}\right)^{-\tau} \right) \\ &= \sum_{i=1}^{p} X_{n,i}(q-1,\tau). \end{aligned}$$

Let $\tau < \min_{i=1,\ldots,p} \tau_i$. Then $X_{n,i}(q-1,\tau) < 0$ for all i, and there exist p constants $c_i > 0$ such that, for large n,

$$X_{n,i}(q-1,\tau) < \exp(-\lambda_n c_i)$$

and hence, for large n,

$$X_n(q-1,\tau) \leq p\exp(-\lambda_n c),$$

where $c := \min_{i=1,\dots,p} c_i$. Therefore

$$X(q-1,\tau) \leq -c < 0,$$

which gives $\tau < \tau_c(q)$, and then

$$\tau_c(q) \geq \min_{i=1,\dots,p} \tau_i(q).$$

Let us prove the opposite inequality. We have

$$\forall i \qquad X_{n,i}(q-1,\tau) \leq X_n(q-1,\tau)$$

and thus

$$\forall i \qquad X_i(q-1,\tau) \leq X(q-1,\tau).$$

This yields

$$\forall i \qquad \tau_c(q) \leq \tau_i(q) \qquad \text{and hence} \qquad \tau_c(q) \leq \min_{i=1,\dots,p} \tau_i(q). \quad \blacksquare$$

Proof of Proposition 13. Set $I := \{1,\dots,p\}$. Define, for all $i \in I$,

$$\alpha_{n,i}(x) := \frac{\log c_{n,i}(I^n(x))}{\log \mu(I^n(x))}$$

$$\alpha_i(x) := \lim_{n\to+\infty} \alpha_{n,i}(x)$$

$$\alpha_n(x) := \frac{\log c_n(I^n(x))}{\log \mu(I^n(x))}$$

$$\alpha(x) := \lim_{n\to+\infty} \alpha_n(x)$$

$$E_{\alpha,i} := \{x \in [\alpha_i; b_i[, \alpha_i(x) = \alpha\}$$

$$E_\alpha := \{x \in [0;1[, \alpha(x) = \alpha\}$$

$$f_{h,i}(\alpha) := \dim_\mu E_{\alpha,i}$$

$$f_h(\alpha) := \dim_\mu E_\alpha$$

$$K_\varepsilon^{n,i}(\alpha) := \{k, \alpha_{n,i}(x_k^n) \in [\alpha-\varepsilon; \alpha+\varepsilon[\}$$

$$K_\varepsilon^n(\alpha) := \{k, \alpha_n(x_k^n) \in [\alpha-\varepsilon; \alpha+\varepsilon[\},$$

where x_k^n is the lower bound of I_k^n. Notice that

$$\forall i, \forall x \in [a_i; b_i[\qquad \alpha(x) = \alpha_i(x).$$

A straightforward calculation shows that

$$E_\alpha = \bigcup_{i \in I} E_{\alpha, i},$$

which gives

$$f_h(\alpha) := \dim_\mu E_\alpha = \sup_{i \in I} f_{h,i}(\alpha)$$

and, using $f_{l,i} = f_{h,i}^{**}$,

$$\forall i \in I \qquad f_h^* \leq f_{h,i}^* = \tau_i$$

($\tau_i = \tau_i^{**}$ because τ_i is concave), or equivalently

$$f_h^* \leq \min_{i \in I} \tau_i = \tau.$$

We thus obtain

$$f_l = \tau^* \leq f_h^{**}$$

and

$$f_l = f_h^{**}$$

Let us prove $f_{l,c_i} = \tilde{f}_{g,c_i}^{**} \Rightarrow f_{l,c} = \tilde{f}_{g,c}^{**}$. One easily shows that

$$\forall \varepsilon > 0, \forall i, \forall \alpha \qquad K_\varepsilon^{n,i}(\alpha) \subset K_\varepsilon^n(\alpha).$$

Thus

$$S_\varepsilon^{n,i}(\alpha, \beta) := \sum_{k \in K_\varepsilon^{n,i}(\alpha)} \mu(I_k^n)^\beta \leq \sum_{k \in K_\varepsilon^n(\alpha)} \mu(I_k^n)^\beta =: S_\varepsilon^n(\alpha, \beta),$$

which implies

$$\forall \varepsilon > 0, \forall i, \forall \alpha \qquad \tilde{f}_{g,i}^\varepsilon(\alpha) \leq \tilde{f}_g^\varepsilon(\alpha)$$

and

$$\forall i, \forall \alpha \qquad \tilde{f}_{g,i}(\alpha) \leq \tilde{f}_g(\alpha).$$

We then obtain

$$\forall i \qquad \tilde{f}_g^* \leq \tilde{f}_{g,i}^* = \tau_i$$

and

$$\tilde{f}_g^* \leq \min_{i \in I} \tau_i = \tau .$$

This yields

$$f_l := \tau^* \leq \tilde{f}_g^* .$$

We conclude that

$$f_l = \tilde{f}_g^{**} . \qquad \blacksquare$$

REFERENCES

1. B. B. Mandelbrot, Possible refinement of the lognormal hypothesis concerning the distribution of energy dissipation in intermittent turbulence, *in* ''Statistical Models and Turbulence (La Jolla, CA 1972)'' (M. Rosenblatt and C. Van Atta, Eds.), pp. 331–351, Lecture Notes in Physics, Vol. 12, Springer-Verlag, New York, 1972.
2. B. B. Mandelbrot, Intermittent turbulence in self similar cascades: Divergence of high moments and dimension of the carrier, *J. Fluid Mech.* **62** (1974), 331–358.
3. B. B. Mandelbrot, Multifractal measures, especially for the geophysicist, *Pure Appl. Geophys.* **131** (1989), 5–42.
4. B. B. Mandelbrot, A class of multinomial multifractal measures with negative (latent) values for the dimension $f(\alpha)$, *in* ''Fractals' Physical Origin and Properties (Erice, 1988)'' (L. Pietronero, Ed.), pp. 3–29, Plenum, New York, 1989.
5. B. B. Mandelbrot, Limit lognormal multifractal measures, *in* ''Frontiers of Physics; Landau Memorial Conference, (Tel Aviv, 1988)'' (E. A. Gostman *et al.*, Eds.), No. 163, pp. 309–340, Pergamon, Elmsford, NY, 1990.
6. U. Frisch and G. Parisi, *in* ''Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics'' (M. Ghil, R. Benzi, and G. Parisi, Eds.), p. 84, North-Holland, Amsterdam, 1985.
7. T. C. Hasley, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Fractal measures and their singularities: The characterization of strange sets, *Phys. Rev.* (February 1986).
8. B. B. Mandelbrot, Multiplications aléatoires itérées et distributions invariantes par moyenne pondérée aléatoire, *C.R. Acad. Sci. Paris Ser. A* **278** (1974), 289–292.
9. B. B. Mandelbrot, Multiplications aléatoires itérées et distributions invariantes par moyenne pondérée aléatoire: Quelques extensions, *C.R. Acad. Sci. Paris Ser. A* **278** (1974), 355–358.
10. B. B. Mandelbrot, Random multifractals: Negative dimensions and the resulting limitations of the thermodynamical formalism, *Proc. Roy. Soc. London Ser. A* **434** (1991), 79–88.
11. C. J. G. Evertsz and B. B. Mandelbrot, Multifractal measures, *in* ''Chaos and Fractals: New Frontiers in Science'' (H.-O. Peitgen, H. Jürgens, and D. Saupe, Eds.), pp. 849–881, Springer-Verlag, New York, 1992.

12. B. B. Mandelbrot, Negative dimensions and Hölders, multifractals, and the role of lateral preasymptotics in science, *in* ‘‘Fourier Analysis and Applications, Paris, 1993 (J. P. Kahane Meeting)’’ (J. Peyrière, Ed.).
13. K. J. Falconer, The multifractal spectrum of statistically self-similar measures, *Journal of Theoret. Probab.* **7**, No. 3 (1994), 681–702.
14. G. Brown, G. Michon, and J. Peyrière, On the multifractal analysis of measures, *J. Statist. Phys.* **66** (1992), 775–790.
15. R. Holley and E. Waymire, Multifractal dimensions and scaling exponents for strongly bounded random cascades, *Ann. Appl. Probab.* **2**, No. 4 (1992), 819–845.
16. M. Arbeiter and N. Patzschke, Random self-similar multifractal, *Adv. in Math.*, to appear.
17. P. Deheuvels, Fractales aléatoires générées par des processus empiriques, *Ann. Probab.* (1994).
18. L. Olsen, ‘‘Random Geometrically Graph Directed Self-Similar Multifractals,’’ Pitman Research Notes in Mathematics, Vol. 307, Pitman, London, 1994.
19. E. Bacry, J.-F. Muzy, and A. Arneodo, Singularity spectrum of fractal signal from wavelet analysis: Exact results, *J. Statist. Phys.*, to appear.
20. S. Jaffard, Construction de fonctions multifractales ayant un spectre de singularites prescrit, *C.R. Acad. Sci. Paris Ser. I* **315** (1992), 19–24.
21. J. Lévy Véhel and P. Mignot, Multifractal segmentation of images, *Fractals* **2**, No. 3 (1994), 371–377.
22. J. Lévy Véhel and J.-P. Berroir, Image analysis through multifractal description, *in* ‘‘Fractal, 1993, Londres, 1993.’’
23. C. Dellacherie, ‘‘Capacités et processus stochastiques,’’ Springer-Verlag, Berlin/New York, 1972.
24. P. Billingsley, Ergodic Theory and Information,’’ Krieger, Melbourne, FL, 1978.
25. J. Peyrière, Calculs de dimensions de Hausdorff, *Duke Math. J.* **44**, No. 3 (1977), 591–601.
26. R. Riedi, An improved multifractal formalism and self-similar measures, *J. Math. Anal. Appl.*, to appear.
27. G. Pólya and G. Szegö, ‘‘Problems and Theorems in Analysis,’’ Vol. I, 1972.